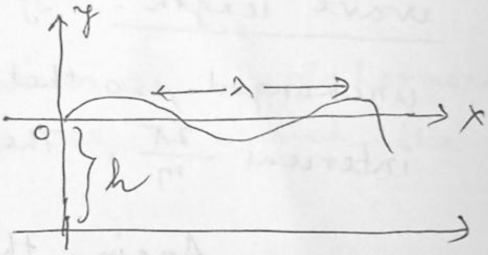
(a) <u>Tidal waves:</u>

Such waves aries when the depth of the liquid is small compard to the wave length and the disturbance affects the motion of the whole liquid.

In this waves, the vertical acceleration of the liquid is negligible as compared to the horizontal acceleration and the plane of the liquid moves as a whole.

(b) <u>Surface waves:</u>



Such waves occure when the wave length of the oscillation is small compared to the depth of the liquid and hence the disturbance does not extend for below the surface.. In these waves, the vertical acceleration is applicable and so it cannot be neglected wind waves and surface tension waves are examples of surface waves. Such waves occur in deep and unbounded (in horizontal direction) liquids like lakes and occurs.

Deduction of the equation of long waves:

Waves whose slope is gradual and whose wave length λ is large compared to the depth $h(x)$ of the liquid is called long wave. The equation of the long wave is obtained under the following assumptions.

(1) The surface elevation η is small with its derivatives so squares and product of them are negligible.

(2) The velocity of the particles is small with its derivatives so squares and products of them are negligible.

(3) The vertical acceleration of the liquid particles is negligible

Let us consider a section of the liquid in a vertical plane in a channel with vertical walls. We take x-axis along the horizontal direction and y-axis vertical upwards with origin at the mean surface (sometime at bottom surface also).

Let u, v be the velocity components along the x and y-axis respectively and p is the pressure, all are functions of x, y and t, also ρ is a uniform constant density. The motion of the liquid is governed by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(Equation of continuity)} \longrightarrow (1)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{(Equation of motion along } x\text{-direction)} \longrightarrow (2)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g$$ (Eqⁿ of motion along y-direction) $\rightarrow$ (3) (iii)

and the condition of irrotationality

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \longrightarrow (4) \text{ (iv)}$$

with the assumption (3) we get, from equation (iii)

$$\frac{1}{\rho}\frac{\partial p}{\partial y} + g = 0 \rightarrow \text{(v)}$$

Integrating this w.r.to y, we get,

$$p + \rho g y = \text{constant} \longrightarrow \text{(vi)}$$

It shows that the pressure remains same as hydrostatic pressure for the liquid motion. But the pressure is a constant on the free surface.

Let on the free surface $p = \pi$

$$\therefore p = \pi \text{ on } y = \eta \text{ for all } x \text{ and } t. \rightarrow \text{(vii)}$$

Hence $\pi + \rho g \eta = \text{constant} \longrightarrow \text{(viii)}$

From (vi) and (viii) we have,

$$p - \pi = -\rho g (y - \eta) \longrightarrow \text{(ix)}$$

Differentiating (ix) partially w.r.to 'x', we get,

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \eta}{\partial x}$$

This shows that $\frac{\partial p}{\partial x}$ is a function of x and t only.

Again, from eqⁿ (ii) we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x} \longrightarrow \text{(x)}$$

This gives that acceleration in horizontal direction is independent of y i.e. $u = u(x,t)$

Hence (ix) simplifies to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x} \longrightarrow \text{(xi)}$$

Now, from the equation of continuity (1), we have,

$$v + \int \frac{\partial u}{\partial x}\, dy = \text{constant}.$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \Rightarrow v = -\int \frac{\partial u}{\partial x}\, dy + \text{const.}$$

It shows that v varies linearly with y, the vertical direction.

Besides, $$\int_{-h}^{\eta} \frac{\partial v}{\partial y}\, dy + \int_{-h}^{\eta} \frac{\partial u}{\partial x}\, dy = 0$$

$$\Rightarrow v(\eta) - v(h) + \int_{-h}^{\eta} \frac{\partial u}{\partial x}\, dy = 0 \longrightarrow (XII)$$

Again, $y - \eta = 0$ represents the boundary of the fluid.

$$\therefore \frac{D}{Dt}(y-\eta) = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(y-\eta) = 0$$

$$\Rightarrow v - \frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} = 0 \quad \text{at } y = \eta \longrightarrow (XIII)$$

In the same way $y + h = 0$ represents a boundary surface

$$\frac{D}{Dt}(y+h) = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(y+h) = 0$$

$$\Rightarrow v + u\frac{\partial h}{\partial x} = 0 \quad \text{at } y = -h \longrightarrow (XIV)$$

Using the conditions (XIII) and (XIV) in (XII), we get,

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + u\frac{\partial h}{\partial x} + \int_{-h}^{\eta} \frac{\partial u}{\partial x}\, dy = 0 \longrightarrow (XV)$$

But $$\frac{\partial}{\partial x}\int_{-h}^{\eta} u\,dy = u\frac{\partial \eta}{\partial x} + u\frac{\partial h}{\partial x} + \int_{-h}^{\eta} \frac{\partial u}{\partial x}\, dy$$

$$= u\frac{\partial \eta}{\partial x} + u\frac{\partial h}{\partial x} + \left[-\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} - u\frac{\partial h}{\partial x}\right] - \frac{\partial \eta}{\partial t} \rightarrow (XVI)$$

; using (XV)

Hence, we find from (XI) and (XVI)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x} \longrightarrow (XVII)$$

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}\int_{-h}^{\eta} u\,dy = -\frac{\partial}{\partial x}\left[u(\eta+h)\right], \longrightarrow (XVIII)$$

These two equation represent equation of long waves in shallow water. These equations are non-linear in character.

If we use the assumptions (I) and (II) for infinitesimal water, we get after neglecting the smaller term in (XVII) and (XVIII)

(non-Linear)

Inertia term (II)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

(pressure term)

(viscosity term $\nu = \frac{\mu}{\rho}$)

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \longrightarrow (XIX)$$

and $$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}(uh) \longrightarrow (XX)$$

XIX → w.r.to t

XX → w.r.to x

These are the linear equation. If we elimite η between them, we get,

$$\frac{\partial^2 u}{\partial t^2} = g\frac{\partial^2}{\partial x^2}(uh) \longrightarrow (XXI)$$

In particular, if h be a constant, fluid has horizontal bottom and then the equation (XXI) becomes

$$\frac{\partial^2 u}{\partial t^2} = gh\frac{\partial^2 u}{\partial x^2}$$

$$= c^2\frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = gh$$

$c = \frac{x}{m}$

$c = \sqrt{gh}$

which is standard one-dimensional wave equation where $c = (gh)^{1/2}$ is called the velocity of the wave.

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If we differentiate (XXII) w.r. to x and use the equation of continuity, then we find that v also, satisfies this equation with little manipulation, we get η and ρ also, satisfy this equation.

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