Fractal geometry

Abstract: In the past, mathematics has been concerned largely with sets and functions to which the methods of classical calculus can be applied. Sets of functions that are not sufficiently smooth or regular have tended to be ignored as pathological and not worthy of study. In recent years this attitude has changed. It has been realized that a great deal can be said, and is worth saying, about the mathematics of non smooth sets. Moreover, irregular sets provide a much better representation of many natural phenomena than do the figures of classical geometry. Fractal geometry provides a general framework for the study of such irregular sets.

Now we begin with some irregular sets of fractal geometry.

1. The middle third Cantor set:

The middle third Cantor set is one of the best known and most easily constructed fractals. It is constructed from a unit interval by a sequence of deletion operations.

Let E_0 be the interval $[0,1]$. Let E_1 be the set obtained by deleting the middle third of E_0, so that E_1 consists of the two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Let E_2 be the set obtained by deleting the middle third of ~~e~~ the intervals of E_1, so that E_2 consists of four intervals $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, $[\frac{8}{9}, 1]$. We continue in this way, with E_K obtained by deleting the middle third of each interval in E_{K-1}. Thus E_K consists of 2^K intervals each of lengths 3^{-K}.

The middle third cantor set F consists of the numbers that are in E_K for all K, i.e F is the intersection $\bigcap_{K=0}^{\infty} E_K$. The cantor set F may be thought of as the limit of the sequence of sets E_K as K tends to infinity. It is impossible to draw the set F itself, with its ~~f~~ infinitesimal detail, so picture of F tend to be pictures of one of the E_K. It might appear that we have removed so much of the interval $[0, 1]$ during the construction of F, that nothing remains. ~~It~~ In fact, F is an infinite set, which contains infinitely many numbers in any neighbourhood of each of its points.

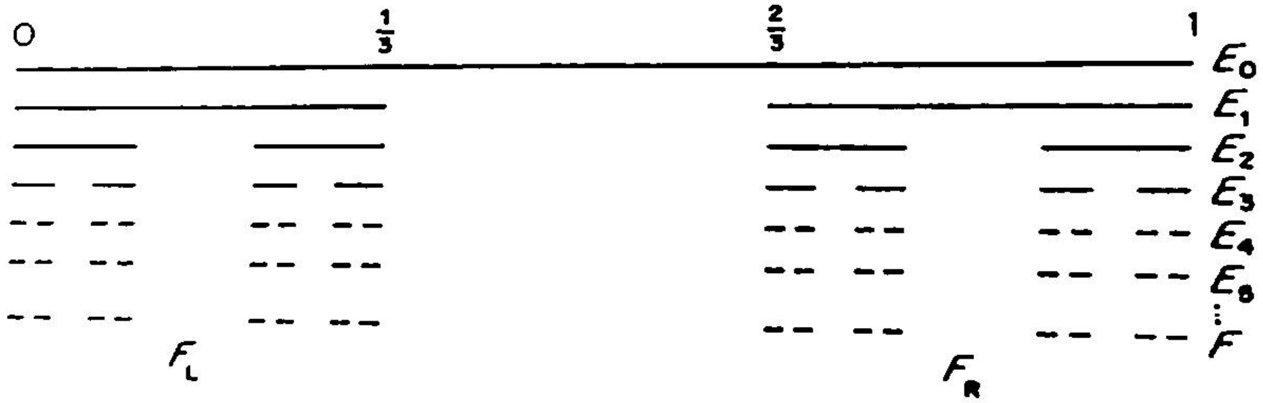

Figure 0.1 Construction of the middle third Cantor set F, by repeated removal of the middle third of intervals. Note that F_L and F_R, the left and right parts of F, are copies of F scaled by a factor $\frac{1}{3}$

2. The von Koch curve:

Let E_0 be a line segment of unit length. The set E_1 consists of four segments obtained by removing the middle third of E_0 and replacing it by the othe two sides of the equilateral triangle based on the removed segment. We construct E_2 by applying the same procedure to each of the segments in E_1, and so on. Thus E_K comes from replacing the middle third of each straight line segment of E_{K-1} by the other two sides of the equilateral triangle. When K is large, the curves E_{K-1} and E_K differ only in fine detail and as K tends to infinity, the sequence of polygonal curves E_K approaches a limiting curve F, called the von Koch curve.

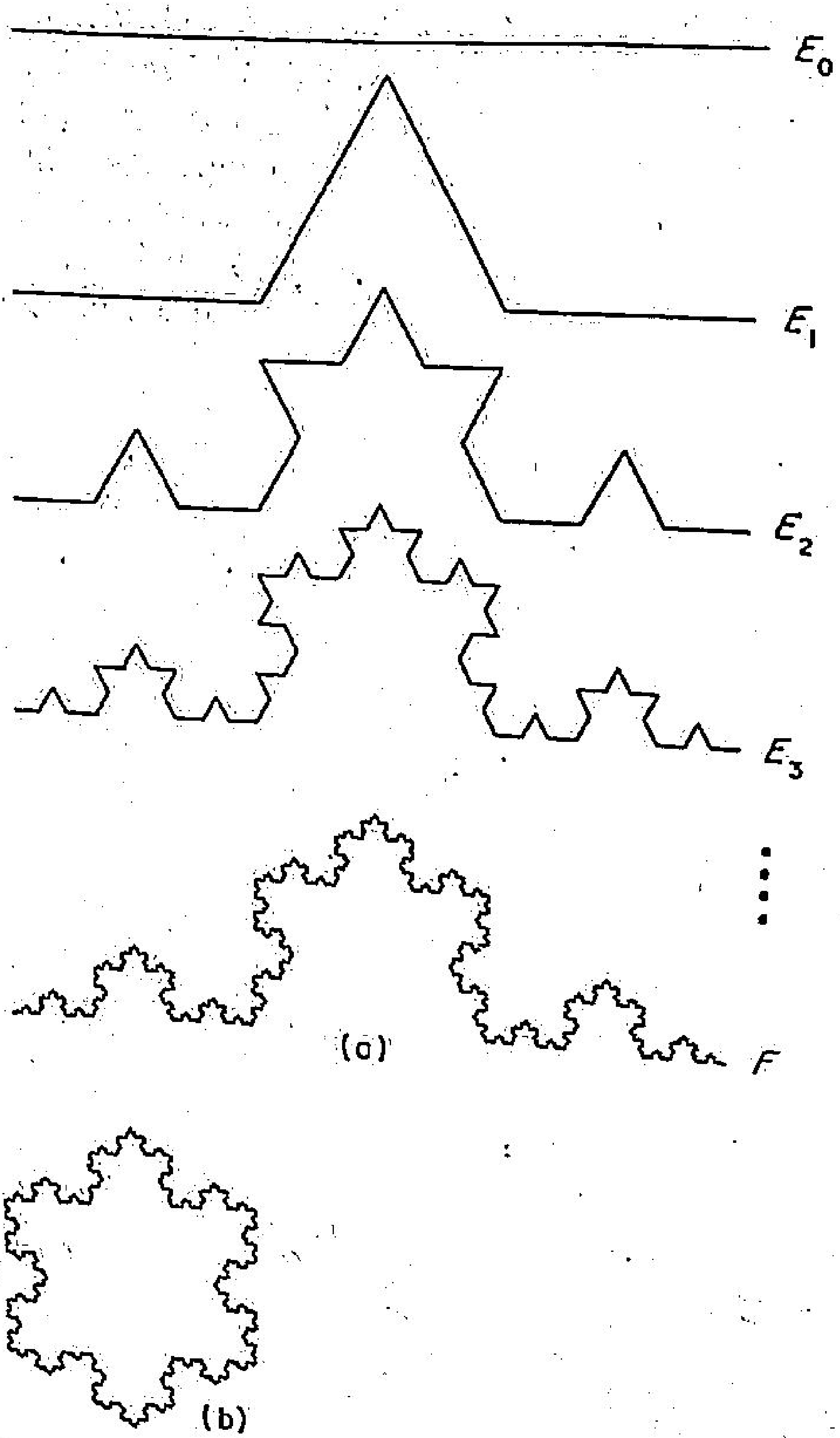

Figure 0.2 (a) Construction of the von Koch curve F. At each stage, the middle third of each interval is replaced by the other two sides of an equilateral triangle. (b) Three von Koch curves fitted together to form a snowflake curve