TENSOR ANALYSIS - II

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TENSORS OF HIGHER RANKS

The laws of transformation of vectors are:

Contravariant … $A'^{\mu} = \partial x'^{\mu} / \partial x^{i} \, A^{i}$ ---- (15)

Covariant … $A'_{\mu} = \partial x^{i} / \partial x'^{\mu} \, A_{i}$ ---- (16)

(a) Contravariant tensors of second rank:

Let us consider $(n)^2$ quantities A^{ij} (here i and j take the values from 1 to n independently) in a system of variables x^i and let these quantities have values $A'^{\mu v}$ in another system of variables x'^{μ}.

If these quantities obey the transformation equations

$$A'^{\mu v} = (\partial x'^{\mu} / \partial x^{i}) \, (\partial x'^{v} / \partial x^{j}) \, A^{ij} \qquad \text{---- (17)}$$

then the quantities A^{ij} are said to be the components of a ***contravariant tensor of second rank***.

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(b) Covariant tensors of second rank:

Let us consider $(n)^2$ quantities A_{ij} (here i and j take the values from 1 to n independently) in a system of variables x^i and let these quantities have values $A'_{\mu v}$ in another system of variables x'^{μ}.

If these quantities obey the transformation equations

$$A'_{\mu v} = (\partial x^{i} / \partial x'^{\mu}) \, (\partial x^{j} / \partial x'^{v}) \, A_{ij} \qquad \text{---- (18)}$$

then the quantities A_{ij} are said to be the components of a ***covariant tensor of second rank***.

(c) Mixed tensor of second rank:

If $(n)^2$ quantities A_j^i in a system of variables x^i are related to another $(n)^2$ quantities A'^{μ}_{v} in another system of variables x'^{μ} by the transformation equations

$$A'^{\mu}_{v} = (\partial x'^{\mu} / \partial x^{i}) (\partial x^{j} / \partial x'^{v}) A_j^i \qquad \ldots (19)$$

then the quantities A_j^i are said to be component of a mixed tensor of second rank.

(d) Tensor of higher ranks:

The tensors of higher ranks are defined by similar laws. The rank of a tensor only indicates the number of indices attached to its per component. For example A_p^{ijk} are the components of a mixed tensor of rank 4; contravariant of rank 3 and covariant of rank 1, if they transform according to the equation

$$A'^{\mu v \sigma}_{q} = (\partial x'^{\mu} / \partial x^{i})(\partial x'^{v} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k}) (\partial x^{p} / \partial x'^{q}) A_p^{ijk} \quad \ldots (20)$$

RANK OF A TENSOR

The rank of a tensor when raised as power to the number of dimensions gives the number of components of the tensor. For example 'a tensor of rank r' in n dimensional space has $(n)^r$ components. Thus the rank of a tensor gives the number of the mode of changes of a physical quantity when passing from one system to the other which is in rotation relative to the first.

Obviously a quantity that remains unchanged when axes are rotated is a tensor of zero rank. The tensors of zero rank are scalars or invariant and similarly the tensors of rank one are vectors

SYMMETRIC AND ANTISYMMETRIC TENSORS

(a) Symetric Tensor:

If two contravariant or covarint indices can be interchanged without altering the tensor, then the tensor is said to be symmetric with respect to these two indices.

For example if $A^{ij} = A^{ji}$

or $A_{ij} = A_{ji}$ ….. (21)

then the contracariant tensor of second rank A^{ij} or covariant tensor A_{ij} is said to be symmetric.

For a tensor A_l^{ijk} of higher rank

if $A_l^{ijk} = A_l^{jik}$

then the tensor A_l^{ijk} is said to be symmetric with respect to indices i and j.

Theorem 1: The symmetry property of a tensor in independent of co-ordinate system used.

If tensor A_l^{ijk} is symmetric with respect to first indices i and j, we have $A_l^{ijk} = A_l^{jik}$ ---- (22)

Now $A'^{\mu\nu\sigma}_{p} = (\partial x'^{\mu} / \partial x^{i})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_l^{ijk}$

$= (\partial x'^{\mu} / \partial x^{i})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_l^{jik}$

[using eq (22)]

Again interchanging the dummy indices i and j, we get

$A'^{\mu\nu\sigma}_{p} = (\partial x'^{\mu} / \partial x^{j})(\partial x'^{\nu} / \partial x^{i})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_l^{ijk}$

$= (\partial x'^{\nu} / \partial x^{i})(\partial x'^{\mu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_l^{ijk}$

$= A'^{\nu\mu\sigma}_{p}$

i.e. given tensor is gain symmetric with respect to first two indices in new co-ordinate system. Thus the symmetry property of a tensor is independent of coordinate system.

Theorem 2: Symmetry is not preserved with respect to two indices, one contravarient and the other covariant.

Let A_l^{ijk} be symmetric with respect to two indices, one contravarient i and the other covariant l, then we have

$$A_l^{ijk} = A_i^{ljk} \quad \text{- ---- (23)}$$

Now $A'^{\mu\nu\sigma}_{p} = (\partial x'^{\mu} / \partial x^{i})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_l^{ijk}$

$= (\partial x'^{\mu} / \partial x^{i})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{p})\, A_i^{ljk}$

[using eq (23)]

Again interchanging the dummy indices i and l, we get

$$A'^{\mu\nu\sigma}_{p} = (\partial x'^{\mu} / \partial x^{l})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{i} / \partial x'^{p})\, A_i^{ljk}$$

$$A'^{p\nu\sigma}_{\mu} = (\partial x'^{p} / \partial x^{i})(\partial x'^{\nu} / \partial x^{j})(\partial x'^{\sigma} / \partial x^{k})(\partial x^{l} / \partial x'^{\mu})\, A_l^{ijk}$$

Thus $A'^{\mu\nu\sigma}_{p} \neq A'^{p\nu\sigma}_{\mu}$

(b) Antisymmetric tensors or skew symmetric tensors.

A tensor, whose each component alters in sign but not in magnitude when two contravariant or covariant indices are interchanged, is said to be skew symmetric or antisymmetric with respect to these two indices.

For example if $A^{ij} = -A^{ji}$

or $A_{ij} = -A_{ji}$ ….. . (24)

then contravariant tensor A^{ij} or covariant tensor A_{ij} of second rank is antisymmetric or for a tensor of higher rank A_l^{ijk}

if $A_l^{ijk} = -A_l^{ikj}$ then tensor A_l^{ijk} is antisymmetric with respect to indices j and k.

The skew-symmetry property of a tensor is also independent of the choice of coordinate system. So if a tensor is skew symmetric with respect to two indices in any coordinate system, it remains skew-symmetric with respect to these two indices in any other co-ordinate system.

If all the indices of a contravariant or covariant tensor can be interchanged so that its components change sign at each interchange of a pair of indices, the tensor is said to be antisymmetric,

i.e., $A^{ijk} = - A^{jik} = + A^{jki}$.

Thus we may state that a contravariant or covariant tensor is antisymmetric if its components change sign under an odd permutation of its indices and do not change sign under an even permutation of its indices

ALGEBRAIC OPERATIONS ON TENSORS

(i) Additional and subtraction: The addition and subtraction of tensors is defined only in the case of tensors of **same rank and same type**. Same type means the same number of contravarient and covariant indices. The addition or subtraction of two tensors, like vectors, involves the individual elements. To add or subtract two tensors the corresponding elements are added or subtracted.

The sum or difference of two tensors of the same rank and same type is also a tensor of the same rank and same type.

For example if there are two tenors ${A_k}^{ij}$ and ${B_k}^{ij}$ of the same rank and same type, then the laws of addition and subtraction are given by

$${A_k}^{ij} + {B_k}^{ij} = {C_k}^{ij} \quad \text{(Addition)} \quad \ldots .(25)$$

$${A_k}^{ij} - {B_k}^{ij} = {D_k}^{ij} \quad \text{(Subtraction)} \ldots . (26)$$

where ${C_k}^{ij}$ and ${D_k}^{ij}$ are the tensors of the same rank and same type as the given tensors.

The transformation laws for the given tensors are

$$A'^{\mu\nu}_{\sigma} = (\partial x'^{\mu}/\partial x^{i})(\partial x'^{v}/\partial x^{i})(\partial x^{k}/\partial x'^{\sigma})A_{k}^{ij} \quad \ldots .(27)$$

and $$B'^{\mu\nu}_{\sigma} = (\partial x'^{\mu}/\partial x^{i})(\partial x'^{v}/\partial x^{i})(\partial x^{k}/\partial x'^{\sigma})B_{k}^{ij} \quad \ldots .(28)$$

Adding (27) and (28), we get

$$A'^{\mu\nu}_{\sigma} + B'^{\mu\nu}_{\sigma} = (\partial x'^{\mu}/\partial x^{i})(\partial x'^{v}/\partial x^{i})(\partial x^{k}/\partial x'^{\sigma})(A_{k}^{ij} + B_{k}^{ij})$$

$$C'^{\mu\nu}_{\sigma} = (\partial x'^{\mu}/\partial x^{i})(\partial x'^{v}/\partial x^{i})(\partial x^{k}/\partial x'^{\sigma})C_{k}^{ij}$$

where is a transformation law for the sum and is similar to transformation laws for A_{k}^{ij} and B_{k}^{ij} given by (27) and (28).

Hence the sum $C_{k}^{ij} = A_{k}^{ij} + B_{k}^{ij}$ is itself a tensor of the same rank and same type as the given tensors.

(ii) Outer product:

The outer product of two tensors is a tensor whose rank is the sum of the ranks of given tensors.

Thus if r and $r^/$ are the ranks of two tensors, their outer product will be a tensor of rank (r + $r^/$).

For example if A_k^{ij} and B_m^{l} are two tensors of ranks 3 and 2 respectively, then $A_k^{ij}B_m^{l} = C_{km}^{ijl}$ (say) …(29)

C_{km}^{ijl} is a tensor of rank 5 (= 3 + 2)

<u>Prove</u>:- If A_k^{ij} and $B_m^{\ l}$ are two tensors of ranks 3 and 2 respectively, then $A_k^{ij}B_m^{\ l} = C_{km}^{\ \ ijl}$ is a tensor of rank 5.

<u>Proof</u>:- The transformation equations of the given tensors as

$$A'^{\ \mu\nu}_{\sigma} = (\partial x'^{\mu}/\partial x^{i})(\partial x'^{\nu}/\partial x^{i})(\partial x^{k}/\partial x'^{\sigma})A_k^{ij} \quad \ldots(30)$$

$$B'^{\ \rho}_{\lambda} = (\partial x'^{\rho}/\partial x^{l})(\partial x^{m}/\partial x'^{\lambda})B_m^{\ l} \quad . \ldots(31)$$

Multiplying (30) and (31), we get

$$A'^{\ \mu\nu}_{\sigma}\, B'^{\ \rho}_{\lambda} =(\partial x'^{\mu}/\partial x^{i})(\partial x'^{\nu}/\partial x^{j})(\partial x^{k}/\partial x'^{\sigma})(\partial x'^{\rho}/\partial x^{l})(\partial x^{m}/\partial x'^{\lambda})A_k^{ij}B_m^{\ l}$$

$$C'^{\ \mu\nu\rho}_{\sigma\lambda}=(\partial x'^{\mu}/\partial x^{i})(\partial x'^{\nu}/\partial x^{j})(\partial x^{k}/\partial x'^{\sigma})(\partial x'^{\rho}/\partial x^{l})(\partial x^{m}/\partial x'^{\lambda})C_{km}^{\ \ ij} \ldots(32)$$

which is a transformation law for tensor of rank 5.
Hence the outer product of two tensors A_k^{ij} and $B_m^{\ l}$ is a tensor $C_{km}^{\ \ ijl}$ of rank 5.

(iii) Contraction of tensors:

The algebraic operation by which the rank of a mixed tensor is lowered by 2 is known as contraction.

In the process of contraction one contravariant index and one covariant index of a mixed tensor are set equal and the repeated index summed over, the result is a tensor of rank lower by two than the original tensor.

For example, consider a mixed tensor ${A_{lm}}^{ijk}$ of rank 5. If we consider k = m, then the tensor become

$${A_{lm}}^{ijm} = A_l^{ij} \quad \text{a tensor of rank 3.}$$

Let us consider a mixed tensor $A_{lm}{}^{ijk}$ of rank 5 with contravariant indices i, j, k and covariant indices l, m.

The transformation law of the given tensor is

$$A'_{\rho\lambda}{}^{\mu\nu\sigma} = (\partial x'^{\mu}/\partial x^{i})(\partial x'^{\nu}/\partial x^{j})(\partial x'^{\sigma}/\partial x^{k})$$
$$(\partial x^{l}/\partial x'^{\rho})(\partial x^{m}/\partial x'^{\lambda})A_{lm}{}^{ijk} \quad \ldots(33)$$

To apply the process of contraction, we put $\lambda = \sigma$ and obtain

$$A'_{\rho\sigma}{}^{\mu\nu\sigma} = (\partial x'^{\mu}/\partial x^{i})(\partial x'^{\nu}/\partial x^{j})(\partial x'^{\sigma}/\partial x^{k})$$
$$(\partial x^{l}/\partial x'^{\rho})(\partial x^{m}/\partial x'^{\sigma})A_{lm}{}^{ijk}$$
$$= (\partial x'^{\mu}/\partial x^{i})(\partial x'^{\nu}/\partial x^{j})(\partial x^{l}/\partial x'^{\rho})\ \delta_{k}{}^{m}A_{lm}{}^{ijk}$$
$$= (\partial x'^{\mu}/\partial x^{i})(\partial x'^{\nu}/\partial x^{j})(\partial x^{l}/\partial x'^{\rho})\ A_{lk}{}^{ijk}$$
$$= (\partial x'^{\mu}/\partial x^{i})(\partial x'^{\nu}/\partial x^{j})(\partial x^{l}/\partial x'^{\rho})\ A_{lk}{}^{ijk}$$
$$[\text{since } (\partial x'^{\sigma}/\partial x^{k})(\partial x^{m}/\partial x'^{\sigma}) = \delta_{k}{}^{m}]$$

which is a transformation law for a mixed tensor of rank 3. Hence $A_{lk}{}^{ijk}$ is a mixed tensor of rank 3 and may be denoted by $A_{l}{}^{ij}$.

Thus the process of contraction enables us to obtain a tensor of rank (r – 2) from a mixed tensor of rank r.

(iv) Inner Product:

The outer product of two tensors followed by a contraction results a new tensor called and inner product of the two tensors and the process is called the inner multiplication of two tensors.

For example, Consider two tensors A_k^{ij} and B_m^{l}.

The outer product of these two tensors is $A_k^{ij}B_m^{l}$ = C_{km}^{ijl} (say).

Applying contraction process by setting m = i, we obtain

$$A_k^{ij}B_m^{l} = C_{km}^{ijl} = D_k^{jl} \qquad \text{(a new tensor)} \qquad \ldots\ldots (34)$$

The new tensor D_k^{jl} is the inner product of the two tensors A_k^{ij} and B_m^{l} .

Prove: The inner product of two tensors of rank one is an invariant.

Proof: Let us consider two tensors of rank 1 as A^i and B_j .

The outer product of A^i and B_j is $A^i\, B_j$ = $C_j^{\,i}$.

Applying contraction process by setting i = j,

we get $A^i B_j$ = C^i_i (a scalar or a tensor of rank zero).

Thus the inner product of two tensors of rank one is a tensor of rank zero. (i.e., invariant). *proved*

(v) Extension of rank:

The rank of a tensor can be extended by differentiating its each component with respect to variables x^i .

Let us consider a simple case in which the original tensor is of rank zero, i.e., a scalar S (x^i) whose, derivatives relative to the variables x^i are $\partial S/\partial x^i$.

In other system of variables $x^{/\mu}$ the scalar is $S^{/}(x^{/\mu})$, such that

$$\partial S/\partial x^{/\mu} = (\partial S/\partial x^i)(\partial x^i/\partial x^{/\mu}) = (\partial x^i/\partial x^{/\mu})\ \partial S/\partial x^i \quad \text{……(35)}$$

This shows that $\partial S/\partial xi$, transforms like the components of a tensor of rank one. Thus the differentiation of a tensor of rank zero gives a tensor of rank one.

In general we may say that the differentiation of a tensor with respect to variables x^i gives a new tensor of rank one greater than the original tensor.

Thanks