Theorem 2: If the intrinsic derivative of a vector along a curve C vanishes at all points of C, then the magnitude of the vector is constant along the curve.

Proof: Let A^i be a vector such that its intrinsic derivative along a curve C vanishes at all points of C. Then —

$$A^i_{,l} \frac{dx^l}{ds} = 0 \text{ at all points of } C$$

Let A be the magnitude of the vector. Then —

$$A^2 = g_{ij} A^i A^j$$

Now, since A^2 is a scalar invariant and the intrinsic derivatives of a scalar invariant is equal to its ~~total~~ derivative, therefore

$$dA^2/ds = A^2_{,l} \frac{dx^l}{ds}$$

$$= (g_{ij} A^i A^j)_{,l} \frac{dx^l}{ds}$$

$$= (g_{ij} (A^i A^j)_{,l} + g_{ij,l} (A^i A^j)) \frac{dx^l}{ds}$$

$$A^2_{,l} \frac{dx^l}{ds} = \frac{dA^2}{ds}$$

$$= g_{ij} (A^i A^j)_{,l} \frac{dx^l}{ds} \quad , \quad \because g_{ij,l} = 0$$

$$= g_{ij} (A^i A^j_{,l} + A^i_{,l} A^j) \frac{dx^l}{ds}$$

$$= g_{ij} A^i A^j_{,l} \frac{dx^l}{ds} + g_{ij} A^i_{,l} A^j \frac{dx^l}{ds}$$

$$= g_{ij} A^j \left(A^i_{,l} \frac{dx^l}{ds} \right) + g_{ij} \left(A^i_{,l} \frac{dx^l}{ds} \right) A^j$$

$= 0 + 0$ at all points of C, by (1)

$= 0$

$\therefore A^2 =$ constant along the curve C

Hence the magnitude of the vector A^i is constant along the curve.

Theorem 3: A vector of constant magnitude is orthogonal to its intrinsic derivative in any direction.

Proof: Let A^i be a vector of constant magnitude. Then $g_{ij} A^i A^j = \text{Const.}$

$$\Rightarrow \frac{d}{ds} (g_{ij} A^i A^j) = 0$$

Now, since the magnitude of the vector is a scalar invariant, therefore —

$$(g_{ij} A^i A^j)_{,l} \frac{dx^l}{ds} = \frac{d}{ds} (g_{ij} A^i A^j)$$

$$\Rightarrow (g_{ij} A^i A^j)_{,l} \frac{dx^l}{ds} = 0, \text{ by (1)}$$

$$\Rightarrow (g_{ij} (A^i A^j)_{,l} + g_{ij,l} (A^i A^j)) \frac{dx^l}{ds} = 0$$

$$\Rightarrow g_{ij} (A^i A^j)_{,l} \frac{dx^l}{ds} = 0, \quad \because g_{ij,l} = 0$$

$$\Rightarrow g_{ij} (A^i_{,l} A^j + A^i A^j_{,l}) \frac{dx^l}{ds} = 0$$

$$\Rightarrow g_{ij} A^i_{,l} A^j \frac{dx^l}{ds} + g_{ij} A^i A^j_{,l} \frac{dx^l}{ds} = 0$$

$$\Rightarrow g_{ij} A^j \left(A^i_{,l} \frac{dx^l}{ds} \right) + g_{ji} A^j \left(A^i_{,l} \frac{dx^l}{ds} \right) = 0$$

$$\Rightarrow A_i \left(A^i_{,l} \frac{dx^l}{ds} \right) + A_i \left(A^i_{,l} \frac{dx^l}{ds} \right) = 0$$

$$\Rightarrow 2A_i \left(A^i_{,l} \frac{dx^l}{ds} \right) = 0$$

$$\Rightarrow A_i \left(A^i_{,l} \frac{dx^l}{ds} \right) = 0$$

This shows that the vector A^i of constant magnitude is orthogonal to its intrinsic derivative in any direction.