Theorem 4 : If the intrinsic derivatives of two unit vectors along a given curve C vanish at all points of C, then the vectors are inclined at a constant angle.

Proof : Let a^i and b^j be two unit vectors such that their intrinsic derivatives along a curve C vanish at all points of C.

Then $a^l_{,l} \frac{dx^l}{ds} = 0$ and $b^j_{,l} \frac{dx^l}{ds} = 0$ at all points of C.

Let the vectors a^i and b^j be inclined at angle θ. Then

$$\cos\theta = g_{ij}\, a^i b^j$$

$$\Rightarrow \frac{d}{ds}(\cos\theta) = \frac{d}{ds}(g_{ij}\, a^i a^j)$$

$$= (g_{ij} a^i b^j)_{,l} \frac{dx^l}{ds} \quad \because g_{ij}\, a^i b^j \text{ is a scalar invariant}$$

$$= (g_{ij}(a^i b^j)_{,l} + g_{ij,l}(a^i b^j)) \frac{dx^l}{ds}$$

$$= g_{ij}(a^i b^j)_{,l} \frac{dx^l}{ds} \quad \because g_{ij,l} = 0$$

$$= g_{ij}\left(a^i_{,l}\, b^j \frac{dx^l}{ds} + g_{ij}\, a^i b^j_{,l} \frac{dx^l}{ds}\right.$$

$$= g_{ij}\, b^j \left(a^i_{,l} \frac{dx^l}{ds}\right) + g_{ij}\, a^i \left(b^j_{,l} \frac{dx^l}{ds}\right)$$

$$= 0 + 0, \text{ at all points of } C, \text{ by (1)}$$

$$= 0$$

$\therefore \frac{d}{ds}(\cos\theta) = 0 \Rightarrow \cos\theta = \text{constant} \Rightarrow \theta = \text{constant}.$

This shown that the vectors a^i and b^i are inclined at a constant angle.

Theorem 5 : A geodesic is an autoparallel curve.

Proof : The differential equation of a geodesic is —

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{ij} \frac{dx^i}{ds} \cdot \frac{dx^j}{ds} = 0$$

Which can be expressed as —

$$\frac{d}{ds}\left(\frac{dx^\alpha}{ds}\right) + \Gamma^\alpha_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0$$

$$\Rightarrow \frac{\partial}{\partial x^j}\left(\frac{dx^\alpha}{ds}\right) \frac{dx^j}{ds} + \Gamma^\alpha_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x^j}\left(\frac{dx^\alpha}{ds}\right) + \Gamma^\alpha_{ij} \frac{dx^i}{ds}\right) \frac{dx^j}{ds} = 0$$

$$\Rightarrow \left(\frac{\partial t^\alpha}{\partial x^j} + \Gamma^\alpha_{ij}\, t^i\right) \frac{dx^j}{ds} = 0$$

$$\Rightarrow t^\alpha_{,j} \frac{dx^j}{ds} = 0$$

where $t^{\alpha} = \dfrac{dx^{\alpha}}{ds}$ is the unit tangent vector to the geodesic. The equation ① shows that the unit tangent vector t^{α} suffers parallel displacement along the geodesic. This establishes that a geodesic is an autoparallel curve.