If we differentiate (xxii) w.r.to x and use the equation of continuity, then we find that v also, satisfies this equation with little manipulation, We get η and ρ also, satisfy this equation.

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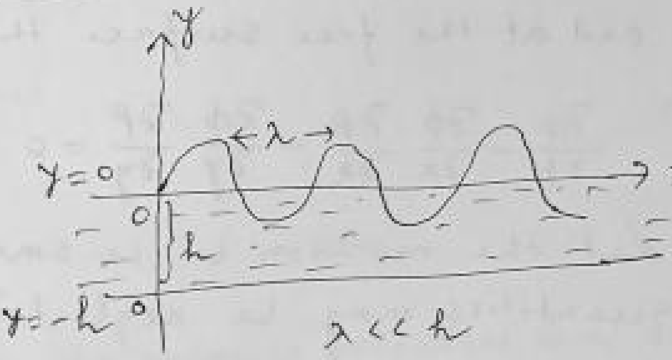

Surface waves:

Surface waves occur when the waves length of the oscillation is much less than the depth of the liquid and the disturbance does not extend far below the surface. for these waves the vertical oscillation is comparable with the horizontal oscillation, and so we consider forces both in horizontal and vertical directions.

Let the x-axis be taken in the undisturbed surface in the direction of propagation of the waves and the y-axis vertically upwards taking the motion to be irrotational, incompressible and two dimensional, the velocity potential ϕ which in turn satisfies Laplace's equation exists, such that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \longrightarrow (1)$$

with the condition $\frac{\partial \phi}{\partial x} = 0 \longrightarrow (11)$

The pressure can be obtained from the Bernoulli's equation,

$$\frac{p}{\rho} + gy - \frac{\partial \phi}{\partial t} = F(t) \longrightarrow (111)$$

Since the free surface is a surface of equipressure $p =$ constant, hence on the free surface

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} = 0 \longrightarrow (IV)$$

where u and v are the velocity components in the directions of x and y respectively. But

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y} \longrightarrow (V)$$

and at the free surface the relation (IV) becomes

$$\frac{\partial p}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial p}{\partial y} = 0 \longrightarrow (VI)$$

Let the motion be so small that the square of the small quantities may be neglected.

Again, without loss of generality we may include $F(t)$ in ϕ and we may take $F(t) = 0$ in (III). Then (III) becomes

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - gy \longrightarrow (VII)$$

Substituting the value of p from (VII) in (VI), we have,

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial t} - \frac{\partial \phi}{\partial y}\cdot\left(\frac{\partial^2 \phi}{\partial y \partial t} - g\right) = 0 \longrightarrow (VIII)$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0 \longrightarrow (IX) \quad \text{(neglecting the smaller terms)}$$

If $\eta(x,t)$ is the elevation of the free surface at time t above the point where abscissa is x, the equation of the free surface is given by

$$y = \eta(x,t) \quad \text{or} \quad y - \eta = 0 \longrightarrow (X)$$

Hence, $\frac{D}{Dt}(y - \eta) = 0$

$$\Rightarrow \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(y - \eta) = 0$$

$$\Rightarrow v - \frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} = 0 \longrightarrow (XI)$$

$$\Rightarrow v = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}$$

Now, $\frac{\partial \eta}{\partial t}$ is η. Again $\frac{\partial \eta}{\partial t}$ being the tangent of the slope of the free surface, so that the second term in (XI) can be neglected and thus we have,

$$v = \frac{\partial \eta}{\partial t} = \eta = -\frac{\partial \phi}{\partial y} \longrightarrow \text{(XII)}$$

which holds at free surface.

Thus we may write

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \phi}{\partial y} \quad \text{at } y=0$$

Thus for the surface waves the velocity potential is a solution of the Laplace's equations (i) which makes $\frac{\partial \phi}{\partial x} = 0$ as a fluid boundary and satisfies (IX) and (XII) at the free surface of the liquid.

We consider the following cases:

Case(I): <u>Progressive wave on the surface of a canal:</u>

We consider the propagation of simple harmonic waves of the type

$$\eta = a\sin(mx - nt) \longrightarrow (1)$$

at the surface of water of uniform depth h contained in a canal with parallel vertical side walls at right angles to the ridges and hollows.

Let the free surface be given by the plane $y=0$ and the bottom of the canal by the plane $y=-h$. The x-axis is taken in the undisturbed surface $y=0$ in the direction of propagation of the waves. and the y-axis vertically upwards.

Then we have to obtain the velocity potential ϕ of the irrotational two-dimensional motion, satisfying Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \longrightarrow (11)$$

Subject to the boundary conditions

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at } y = -h \longrightarrow \text{(iii)}$$

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = 0 \longrightarrow \text{(iv)}$$

and $$v = \frac{\partial \eta}{\partial t} = -\frac{\partial \phi}{\partial y} \text{ on } y = 0 \longrightarrow \text{(v)}$$

on using (i) the condition (v) gives

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial t}\left[a\sin(mx - nt)\right]$$

$$= -a\cos(mx - nt)(-n)$$

$$= an\cos(mx - nt) \quad \text{at } y = 0.$$

This suggest that, we should take the solution of (ii) in the form,

$$\phi = f(y)\cos(mx - nt) \longrightarrow \text{(vi)}$$

Substituting this in the equation (ii), we get,

$$\frac{\partial^2}{\partial x^2}\left[f(y)\cos(mx - nt)\right] + \frac{\partial^2}{\partial y^2}\left[f(y)\cos(mx - nt)\right] = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left[-mf(y)\sin(mx - nt)\right] + \frac{d^2 f}{dy^2}\left[\cos(mx - nt)\right] = 0$$

$$\Rightarrow \left[-m^2 f(y)\cos(mx - nt)\right] + \frac{d^2 f}{dy^2}\left\{\cos(mx - nt)\right\} = 0$$

$$\Rightarrow \frac{d^2 f}{dy^2} - m^2 f = 0, \quad \because \cos(mx - nt) \neq 0.$$

~~Substituting this in the equation (ii), we get,~~

$$\cancel{\frac{\partial^2}{\partial x^2}\left[f(y)\cos(mx - nt)\right] + \frac{\partial^2}{\partial y^2}\left[f(y)\cos(mx - nt)\right] = 0}$$

The solution of this equation is of the form

$$f(y) = Ae^{my} + Be^{-my}.$$

$\therefore$ from (vi), $\phi = \left(Ae^{my} + Be^{-my}\right)\cos(mx - nt) \longrightarrow$ (vii)

Again, the condition (iii) requires $\frac{\partial\phi}{\partial y} = 0$

i.e. $\left[m\left(Ae^{my} - Be^{-my}\right)\cos(mx - nt)\right]_{y=-h} = 0$

i.e. $Ame^{-mh} - Be^{mh} = 0$

i.e. $Ae^{-mh} = Be^{mh} = \frac{1}{2}D$ (say)

Thus $A = \frac{1}{2}De^{mh}$, $B = \frac{1}{2}De^{-mh}$

$\therefore$ $\phi = \left[\frac{D}{2}e^{mh}e^{my} + \frac{1}{2}De^{-mh}e^{-my}\right]\cos(mx - nt)$

$= \frac{1}{2}D\left[e^{m(h+y)} + e^{-m(h+y)}\right]\cos(mx - nt)$

$= D\cosh m(h+y)\cos(mx - nt)$

Next, putting this value of ϕ in the surface condition (iv) for $y = 0$, we get,

$$\frac{\partial^2\phi}{\partial t^2} + g\left(\frac{\partial\phi}{\partial y}\right) = 0$$

$\Rightarrow D\cosh m(y+h)(-1)(-n)(-n)\cos(mx - nt)$
$+ gDm\sinh m(y+h)\cos(mx - nt) = 0$ at $y = 0$

$\Rightarrow -n^2D\cosh m(y+h)\cos(mx - nt) + gDm\sinh(y+h)\cos(mx - nt) =$ at $y =$

$\Rightarrow n^2 = gm\tanh m(y+h)$ at $y = 0$

$\Rightarrow n^2 = gm\tanh mh \longrightarrow$ (ix)

Now if $c = \frac{n}{m}$ and $\lambda = \frac{2\pi}{m}$ denote the velocity of propagation and the wave length, then it follows that,

$$c^2 = \frac{n^2}{m^2} = \frac{gm\tanh mh}{m^2} = \frac{g}{m}\tanh mh$$

$$\Rightarrow c^2 = \left(\frac{\lambda g}{2\pi}\right) \tanh mh \longrightarrow (x)$$

The constant D in (VIII) can be expressed in terms of the amplitude 'a' of the wave by using ① and (VIII) in (V) i.e.

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \phi}{\partial y} \quad \text{at } y=0$$

$$\Rightarrow a\cos(mx-nt)(-n) = -D\sinh m(y+h) \cdot m\cos(mx-nt) \quad \text{at } y=0$$

$$\Rightarrow \frac{na}{m}\frac{\cos(mx-nt)}{\sinh mh} = D\cos(mx-nt) \quad \text{at } y=0$$

$$\Rightarrow D = \frac{na}{m}\frac{1}{\sinh mh} \quad \text{at } y=0$$

so that

$$\phi = \frac{na}{m}\frac{\cosh m(y+h)}{\sinh mh}\cos(mx-nt) \longrightarrow (xI)$$

But we know that

$$n^2 = mg\tanh mh$$

$$\Rightarrow m = \frac{h^2}{g\tanh mh}$$

$$\therefore \phi = \frac{g\tanh mh}{n^2} \times \frac{na\cosh m(y+h)}{\sinh mh} \cdot \cos(mx-nt)$$

$$= \frac{ga}{n}\frac{\cosh m(y+h)}{\cosh mh}\cos(mx-nt) \longrightarrow (xII)$$

We can find out the velocity components as follows:

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} = na\frac{\cosh m(y+h)}{\sinh mh} \cdot \sin(mx-nt) \\ v &= -\frac{\partial \phi}{\partial y} = -na \cdot \frac{\sinh m(y+h)}{\sinh mh} \cdot \cos(mx-nt) \end{aligned}\right\} \longrightarrow (xIII)$$

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