Necessary and Sufficient conditions for equilibrium of a fluid under the action of forces.

The condition is necessary — The fluid is in equilibrium under the action of a system of forces whose components are X, Y, Z per unit mass at a $P(x,y,z)$ along parallel to axes. If p be the pressure at $P(x,y,z)$ in the fluid and ρ be the density of the fluid, then we have

$$\frac{\partial p}{\partial x} = \rho X \longrightarrow (i)$$

$$\frac{\partial p}{\partial y} = \rho Y \longrightarrow (ii)$$

$$\frac{\partial p}{\partial z} = \rho Z \longrightarrow (iii)$$

Since p is a function of the independent variable x, y and z, we have

$$\frac{\partial^2 p}{\partial y \partial z} = \frac{\partial^2 p}{\partial z \partial y}, \quad \frac{\partial^2 p}{\partial z \partial x} = \frac{\partial^2 p}{\partial x \partial z}, \quad \frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x}$$

$$\frac{\partial^2 p}{\partial y \partial z} = \frac{\partial^2 p}{\partial z \partial y}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial P}{\partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial z}(\rho \cancel{Y})$$

$$\Rightarrow \rho \frac{\partial Z}{\partial y} + Z \frac{\partial \rho}{\partial y} = \rho \frac{\partial Y}{\partial z} + Y \frac{\partial \rho}{\partial z}$$

$$\Rightarrow \rho\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) = Z\frac{\partial \rho}{\partial y} - Y\frac{\partial \rho}{\partial z} \longrightarrow (IV)$$

Simmilarly

$$\rho\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) = X\frac{\partial \rho}{\partial z} - Z\frac{\partial \rho}{\partial x} \longrightarrow (V)$$

and $\rho\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = Y\frac{\partial \rho}{\partial x} - X\frac{\partial \rho}{\partial y} \longrightarrow (VI)$

Multiplying (IV), (V) and (VI) by X, Y, Z and adding we have

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0$$

This is the required necessary condition.

We assume that

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0$$

This show that $X\,dx + Y\,dy + Z\,dz$ is an exact differential

$\Rightarrow \rho(X\,dx + Y\,dy + Z\,dz)$ is an exact differential.

$$\therefore \quad \rho(X\,dx + Y\,dy + Z\,dz) = dp$$

$$= \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$$

equating coefficients of dx, dy, dz on both sides, we get

$$\frac{\partial p}{\partial x} = \rho X, \quad \frac{\partial p}{\partial y} = \rho Y, \quad \frac{\partial p}{\partial z} = \rho Z$$

these are clearly the equations of equilibrium, hence the condition is sufficient also.