

Necessary and sufficient conditions for equilibrium of a fluid under the action of forces.

The condition is necessary — The fluid is in occupied equilibrium under the action of a system of forces whose components are  $\sigma_x, \sigma_y, \sigma_z$  per unit mass at a point  $(x, y, z)$  along parallel to axes of symmetry  $O = (x, y, z)$  of the fluid. If  $P$  be the pressure at  $(x, y, z)$  in the fluid and  $\rho$  be the density of the fluid, then we have

$$\frac{\partial P}{\partial x} = \rho \sigma_x \quad (1)$$

$$\text{Similarly, taking derivative w.r.t. } y \text{ we get} \\ \frac{\partial P}{\partial y} = \rho \sigma_y \quad (2)$$

$$\frac{\partial P}{\partial z} = \rho \sigma_z \quad (3)$$

Since  $P$  is a function of the independent variables  $x, y$ , and  $z$ , we have

$$\frac{\partial^2 p}{\partial y \partial z} = \frac{\partial^2 p}{\partial z \partial y}, \quad \frac{\partial^2 p}{\partial z \partial x} = \frac{\partial^2 p}{\partial x \partial z}$$

$$0 = \left(\frac{xy}{yz} - \frac{xz}{yz}\right) \Sigma + \left(\frac{yz}{xz} - \frac{xy}{xz}\right) X + \left(\frac{xy}{xz} - \frac{yz}{xz}\right) Y$$

$$\frac{\partial^2 p}{\partial y \partial z} = \frac{\partial^2 p}{\partial z \partial y}$$

taking no. of terms  $xby + ybx + xyx$

$$\Rightarrow \frac{\partial}{\partial y} \left( \frac{\partial p}{\partial z} \right) = \frac{\partial}{\partial z} \left( \frac{\partial p}{\partial y} \right)$$

$$\Rightarrow \frac{\partial}{\partial y} (yz) = \frac{\partial}{\partial z} (xy)$$

$$\Rightarrow y \frac{\partial z}{\partial y} + z \frac{\partial y}{\partial y} = y \frac{\partial x}{\partial z} + x \frac{\partial z}{\partial z}$$

$$\Rightarrow \left( \frac{yz}{yz} + \frac{yz}{yz} + \frac{xy}{yz} \right) \left( \frac{\partial y}{\partial z} - \frac{\partial z}{\partial y} \right) = z \frac{\partial y}{\partial z} - y \frac{\partial z}{\partial z} \rightarrow \text{IV}$$

After this  $xb, tb, xb$  to differentiate  $\Rightarrow$  factors

Similarly

$$y \left( \frac{\partial z}{\partial x} - \frac{\partial x}{\partial z} \right) = x \frac{\partial z}{\partial z} = \frac{yz}{xz} \frac{\partial z}{\partial x} \Rightarrow \text{V}$$

$$\text{and } y \left( \frac{\partial x}{\partial y} - \frac{\partial y}{\partial x} \right) = y \frac{\partial x}{\partial x} - x \frac{\partial y}{\partial y} \rightarrow \text{VI}$$

Multiplying  $\text{IV}$ ,  $\text{V}$  and  $\text{VI}$  by  $x, y, z$  and adding we have

$$x \left( \frac{\partial y}{\partial z} - \frac{\partial z}{\partial y} \right) + y \left( \frac{\partial z}{\partial x} - \frac{\partial x}{\partial z} \right) + z \left( \frac{\partial x}{\partial y} - \frac{\partial y}{\partial x} \right) = 0$$

This is the required necessary condition.

We assume that  $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ ,  $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0$$

This show that  $Xdx + Ydy + Zdz$  is an exact differential

$\Rightarrow f(Xdx + Ydy + Zdz)$  is an exact differential.

$$\therefore f(Xdx + Ydy + Zdz) = dP$$

$$\text{① } \frac{\partial P}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x \partial z}$$

equating coefficients of  $dx, dy, dz$  on both sides, we get

$$\frac{\partial P}{\partial x} = f_x, \frac{\partial P}{\partial y} = f_y, \frac{\partial P}{\partial z} = f_z$$

These are clearly the equations of equilibrium, hence from the condition is sufficient also

$$\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial z}\right)z + \left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial y}\right)y + \left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right)x$$

for equilibrium. But we have seen that a sup