Theorem 6) A vector of constant magnitude which suffers parallel displacement along a geodesic is inclined at a constant angle to the geodesic.

Proof: Let A^i be a vector of constant magnitude A which suffers parallel displacement along a geodesic C. Then $A^i_{,l}\frac{dx^l}{ds}=0$ at all points of C

Let t^j be the unit tangent vector to the geodesic C. Then —

$$t^j_{,l}\frac{dx^l}{ds} = \left(\frac{\partial t^j}{\partial x^l}+\Gamma^j_{kl}t^k\right)\frac{dx^l}{ds}$$

$$= \frac{\partial t^j}{\partial x^l}\frac{dx^l}{ds}+\Gamma^j_{kl}t^k\frac{dx^l}{ds}$$

$$= \frac{dt^j}{ds}+\Gamma^j_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} \quad ; \quad \because t^k = \frac{dx^k}{ds}$$

$$= \frac{d^2x^j}{ds^2}+\Gamma^j_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} \quad ; \quad \because t^j = \frac{dx^j}{ds}$$

$\because$ the curve c is a geodesic

$$\therefore \frac{d^2x^j}{ds^2}+\Gamma^j_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds}=0.$$

$$\therefore t^j_{,l}\frac{dx^l}{ds}=0 \text{ at points of } C$$

Now let θ be the angle between the vectors A^i and t^j. Then =

$$\cos\theta = \frac{g_{ij}A^it^j}{A\cdot 1} \Rightarrow A\cos\theta = g_{ij}A^it^j$$

$$\Rightarrow \frac{d}{ds}(A\cos\theta) = \frac{d}{ds}(g_{ij}A^it^j)$$

$$= (g_{ij}A^it^j)_{,l}\frac{dx^l}{ds} \quad (\because g_{ij}A^it^j \text{ is a scalar invariant})$$

$$= \left(g_{ij}(A^it^j)_{,l} + g_{ij,l}(A^it^j)\right)\frac{dx^l}{ds}$$

$$= g_{ij}(A^it^j)_{,l}\frac{dx^l}{ds} \quad , \because g_{ij,l}=0$$

$$= g_{ij}A^i_{,l}t^j\frac{dx^l}{ds} + g_{ij}A^it^j_{,l}\frac{dx^l}{ds}$$

$$= g_{ij}t^j\left(A^i_{,l}\frac{dx^l}{ds}\right)+g_{ij}A^i\left(t^j_{,l}\frac{dx^l}{ds}\right)$$

$= 0 + 0$

$= 0$ $\quad$ ($\because A$ is const.)

$\therefore A \cdot \cos\theta = \text{Constant} \Rightarrow \cos\theta = \text{Constant}$

$\Rightarrow \theta = \text{Constant}$

This shows that the vector A^i of constant magnitude A is inclined at a constant angle to the geodesic.

Theorem 7) If two vectors of constant magnitude undergo parallel displacement along a given curve, then they are inclined at a constant angle.

Proof: Let the two vectors A^i and B^j of constant magnitude A and B ~~respe~~ respectively undergo parallel displacement along a given curve C. Then —

$$A^2 = g_{ik} A^i A^k \quad , \quad B^2 = g_{jk} B^j B^k$$

and $$A^i_{,l} \frac{dx^l}{ds} = 0 \quad , \quad B^j_{,l} \frac{dx^l}{ds} = 0 \quad \text{at all points of } C.$$

Let the two vectors, A^i and B^j be inclined at a constant angle θ. Then —

$$\cos\theta = \frac{g_{ij} A^i B^j}{AB}$$

$$\Rightarrow AB\cos\theta = g_{ij} A^i B^j$$

$$\Rightarrow \frac{d}{ds}(AB\cos\theta) = \frac{d}{ds}(g_{ij} A^i B^j)$$

$$= (g_{ij} A^i B^j)_{,l} \frac{dx^l}{ds} \quad \because g_{ij} A^i B^j \text{ is a scalar constant}$$

$$= \left(g_{ij} (A^i B^j)_{,l} + g_{ij,l} (A^i B^j)\right) \frac{dx^l}{ds}$$

$$= g_{ij} (A^i B^j)_{,l} \frac{dx^l}{ds} \quad \because g_{ij,l} = 0$$

$$= g_{ij} A^i_{,l} B^j \frac{dx^l}{ds} + g_{ij} A^i B^j_{,l} \frac{dx^l}{ds}$$

$$= g_{ij}\left(A^i_{,l} \frac{dx^l}{ds}\right) B^j + g_{ij} A^i \left(B^j_{,l} \frac{dx^l}{ds}\right)$$

$$= 0 + 0 \quad \text{at all points of } C$$

$$= 0$$

$\therefore AB\cos\theta = \text{Constant}$

$\Rightarrow \cos\theta = \text{Constant}$, $\because A, B$ are constants.

$\Rightarrow \theta = \text{constant}$

This shows that the vectors A^i and B^j are inclined at a constant angle.