Case(II): Progressive waves in a deep canal:

If the depth h of the canal is sufficiently large in comparison with λ for e^{-mh} to be neglected, then in case I, we must have $B=0$.

i.e. $\phi = A e^{my} \cos(mx-nt) \longrightarrow (XIV)$

If we put this form of ϕ in the expression

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0 \text{ at } y=0 \text{, then we have}$$

$$-An^2 e^{my}\cos(mx-nt) + gAme^{my}\cos(mx-nt) = 0.$$

i.e. $-An^2\cos(mx-nt) + gAm\cos(mx-nt) = 0$

i.e. $-An^2 + Amg = 0$

$\Rightarrow n^2 = gm$

$\Rightarrow c^2m^2 = gm, \left(c = \frac{n}{m}\right)$

$$\boxed{\Rightarrow c^2 = \frac{g}{m}}$$

or, $c^2 = \frac{g\lambda}{2\pi} \longrightarrow (XV) \quad , \left(\because \lambda = \frac{2\pi}{m}\right)$

We now determine the constant A of (XIV) in terms of the amplitude 'a' of the wave using (V) and (XIV) the boundary condition

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \phi}{\partial y} \text{ on } y=0 \text{ gives}$$

$\therefore -na\cos(mx-nt) = -mAe^{my}\cos(mx-nt)$ on $y=0$

$\Rightarrow na = mA$

$\Rightarrow A = \frac{na}{m}$

and hence $\phi = \frac{na}{m} e^{my} \cos(mx - nt) \longrightarrow (xv)$

Then velocity components of the particles are

$$u = -\frac{\partial \phi}{\partial x} = na\, e^{my} \sin(mx - nt)$$

$$v = -\frac{\partial \phi}{\partial y} = -na\, e^{my} \cos(mx - nt)$$

<u>Remark</u>: Complex potential for a progressive wave in a deep canal.

We have, $\phi = \frac{an}{m} e^{my} \cos(mx - nt) = ac\, e^{my} \cos(mx - nt)$, $\because c = \frac{n}{m}$

Using the relation,

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial x} \text{ we have,}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial x}\left[c a e^{my} \cos(mx - nt)\right]$$

$$= -ca\, e^{my} m \sin(mx - nt)$$

Integrating w.r.to 'y', we have,

$$\psi = -ac \frac{n}{m} e^{my} \sin(mx - nt) \quad , \text{[neglecting the canal of integrate}$$

Hence the complex potential ω is given by

$$\omega = \phi + i\psi$$

$$= ac\, e^{my}\left[\cos(mx - nt) - i \sin(mx - nt)\right]$$

$$= ac\, e^{my} e^{-i(mx - nt)}$$

$$= ac\, e^{my} e^{-imx + int}$$

$$= ac\, e^{my} e^{-imx} e^{int}$$

$$= ac\, e^{-im(x+iy) + int}$$

$$= ac\, e^{-imz + int}$$

$$\Rightarrow W = ac\, e^{-i(mz + nt)}$$

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Energy of progressive waves:

Let the wave profile of the progressive waves at the surface of water of depth h, be of the form

$$\eta = a\sin(mx - nt) \longrightarrow (1)$$

$$\phi = \frac{ga}{n}\,\frac{\cosh m(y+h)}{\cosh mh}\cdot\cos(mx - nt) \longrightarrow (11)$$

Let E be the energy of water between two vertical planes parallel to the direction of propagation at unit distance apart, then the potential energy for a single wave length is given by

$$P.E. = \frac{1}{2} g\rho \int_0^{\lambda} \eta^2\, dx.$$

$$= \frac{1}{2} g\rho \int_0^{\lambda} a^2 \sin^2(mx - nt)\, dx$$

$$= \frac{1}{4} g\rho a^2 \int_0^{\lambda} \left[1 - \cos 2(mx - nt)\right] dx \longrightarrow (1)$$

But $$\int_0^{\lambda} \cos 2(mx - nt)\, dx = \frac{1}{2m}\left[\sin 2(mx - nt)\right]_0^{\lambda}$$

$$= \frac{1}{2m}\left[\sin(2m\lambda - 2nt) - \sin(-2nt)\right]$$

$$= \frac{1}{2m}\left[\sin(4\pi - 2nt) + \sin 2nt\right]; \quad \because \lambda = \frac{2\pi}{m} \Rightarrow m = \frac{2\pi}{\lambda}$$

$$= \frac{1}{2m}\left[-\sin 2nt + \sin 2nt\right]$$

$$= 0$$

From (1),

$$P.E. = \frac{1}{4} g\rho a^2 \int_0^{\lambda} dx$$

$$= \frac{1}{4} g\rho a^2 \left[x\right]_0^{\lambda} = \frac{1}{4} g\rho a^2 \lambda \longrightarrow (11)$$

Also, the K.E. is defined by

$$K.E. = \frac{1}{2}\rho\int_{x=0}^{\lambda}\left(\phi\frac{\partial\phi}{\partial y}\right)_{y=0} dx$$

Now, $$\left[\phi\frac{\partial\phi}{\partial y}\right]_{y=0} = \left[\frac{ga}{n}\cdot\frac{\cosh m(y+h)}{\cosh mh}\cos(mx-nt) \times \frac{gam}{n}\,\frac{\sinh m(y+h)}{\cosh mh}\cos(mx-nt)\right]_{y=0}$$

$$= \frac{g^2a^2m}{n^2}\,\frac{\cosh mh}{\cosh mh}\cdot\cos(mx-nt)\cdot\frac{\sinh mh}{\cosh mh}$$

$$= \frac{g^2a^2m}{n^2}\cos^2(mx-nt)\tanh mh$$

$$= \frac{g^2a^2m}{gm\tanh mh}\times\cos^2(mx-nt)\cdot\tanh mh$$

$$= ga^2\cos^2(mx-nt);$$

$$c^2 = \frac{n^2}{m^2} = \frac{g}{m}\tanh mh$$
$$\Rightarrow n^2 = gm\tanh mh$$

From (III), we have,

$$K.E. = \frac{1}{2}\rho\int_0^{\lambda} ga^2\cos^2(mx-nt)\,dx$$

$$= \frac{1}{2}\rho g a^2\int_0^{\lambda}\frac{1}{2}\left[1+\cos(mx-nt)\right]dx$$

$$= \frac{1}{4}\rho g a^2\lambda$$

Thus the total energy V per unit area of the water surface is given by

$$V = K.E. + P.E.$$

$$= \frac{1}{4}g\rho a^2\lambda + \frac{1}{4}g\rho a^2\lambda$$

$$= \frac{1}{2}g\rho a^2\lambda.$$

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Ex. Prove that the velocity of propagation c of surface waves of length λ in a rectangular canal of depth h is

$$c^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}$$

Deduce from this an expression for the velocity of long waves in shallow water.

Sol. 1st part:

Progressive wave : case (I) in equation (x).

2nd part:

For velocity of long waves in shallow water, the wave length λ is very large in comparision with h i.e.

$$\frac{h}{\lambda} \longrightarrow 0 \Rightarrow \frac{hm}{2\pi} \longrightarrow 0 \Rightarrow mh \longrightarrow 0 \qquad ; \because \lambda = \frac{2\pi}{m}$$

Then $\tanh mh \longrightarrow mh$

$$\therefore c^2 = \frac{g}{m} \tanh mh$$

$$= \frac{g}{m} mh = gh$$

$\therefore$ velocity of propagation is $c = \sqrt{gh}$.

Ex. In the case of surface waves, show that the condition to be satisfied at the free surface by the velocity potential ϕ as

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial y} = 0$$

If waves are propagated on the surface of deep water, show that the velocity of propagation of waves length λ is $\sqrt{\frac{\lambda g}{2\pi}}$.

Sol. Previous article (surface waves).

Ex. Establish the equation for long waves in the form

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}$$

where the symbols have their usual meanings.

Ex. Prove that the velocity of propagation of a wave

$$\eta = a \sin(mx - nt)$$

at the surface of water of uniform depth h is given by $c^2 = \frac{g}{m} \tanh mh$.

Ex. Show that when irrotational waves of length λ are propagated in water of infinite depth, the pressure at any particle of water is the same as it was in the equilibrium position of the particle when water was at rest.

Soln: The governing equation of motion are,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (I)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \longrightarrow (II)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g \longrightarrow (III)$$

Since the motion is irrotational therefore

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \longrightarrow (IV)$$

and there exists velocity potential; ϕ such that

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y} \longrightarrow (V)$$

$\vec{\nabla} \times \vec{q} = 0$, $\vec{q} \equiv (u, v, w)$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = 0$$

From equation (iii) we may have,

$$\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial y}\right) + u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial y} - g \text{ ; using (iv) \& (v).}$$

$$\text{or } -\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(u^2+v^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g$$

Integrating w.r.to y,

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2+v^2\right) = -\frac{1}{\rho}p - gy + F(t)$$

$$\text{or, } -\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 = -\frac{1}{\rho}p - gy + F(t) \quad ; \; q^2 = u^2+v^2$$

$$\text{or, } \frac{p}{\rho} = \frac{\partial \phi}{\partial t} - \frac{1}{2}q^2 - gy + F(t) \longrightarrow \text{(vi)}$$

Since q^2 can be neglected for small velocity and adjusting ϕ in such a way that $F(t)=0$, we may write (vi) as

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - gy \longrightarrow \text{(vii)}$$

Let $\phi = Ae^{my}\cos(mx-nt)$

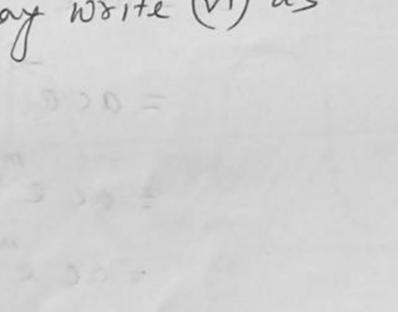
$$\therefore \frac{\partial \phi}{\partial t} = Ame^{my}\sin(mx-nt)$$

when $h \to \infty$ then,

$$\left[\frac{\partial \phi}{\partial t}\right]_{y=-h} = \left[Ane^{my}\sin(mx-nt)\right]_{y=-h}$$

$$= Ane^{-mh}\sin(mx-nt)$$

$$= 0 \quad \text{as } h \to \infty$$



and then $\frac{p}{\rho} = gh$

i.e. $p = \rho gh$

Again, when water is at rest then the pressure at any particle of depth h is given by,

$$p = \rho gh.$$

Proved.