1. INDEFINITE INTEGRAL

INTEGRATION *It is the inverse process of differentiation.*

If the derivative of $F(x)$ is $f(x)$ then we say that the *antiderivative* or *integral* of $f(x)$ is $F(x)$ and we write,

$$\int f(x)\,dx = F(x).$$

Thus, $$\frac{d}{dx}[F(x)] = f(x) \Rightarrow \int f(x)\,dx = F(x).$$

Example Since $\frac{d}{dx}(\sin x) = \cos x$, we have $\int \cos x\,dx = \sin x$.

Moreover, if C is any constant then $\frac{d}{dx}(\sin x + C) = \cos x$.

So, in general, $\int \cos x\,dx = (\sin x + C)$.

Clearly, different values of C will give different integrals.

Thus, a given function may have an indefinite number of integrals. Because of this property, we call these integrals *indefinite integrals.*

Thus, $\frac{d}{dx}[F(x)] = f(x) \Rightarrow \int f(x)\,dx = F(x) + C$, where C is a constant, called the *constant of integration.* Any function to be integrated is known as an *integrand.*

The following two results are a direct consequence of the definition of an integral.

RESULT 1 $\int x^n\,dx = \frac{x^{(n+1)}}{(n+1)} + C$, *when* $n \neq -1$.

PROOF We have, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)\,x^n}{(n+1)} = x^n$.

$$\therefore \quad \int x^n\,dx = \frac{x^{(n+1)}}{(n+1)} + C.$$

Thus, we have

(i) $\int x^6\,dx = \frac{x^{(6+1)}}{(6+1)} + C = \frac{x^7}{7} + C.$

(ii) $\int x^{2/3}\,dx = \dfrac{x^{\left(\frac{2}{3}+1\right)}}{\left(\frac{2}{3}+1\right)} + C = \dfrac{3}{5}x^{5/3} + C.$

(iii) $\int x^{-3/4}\,dx = \dfrac{x^{\left(-\frac{3}{4}+1\right)}}{\left(-\frac{3}{4}+1\right)} = 4x^{1/4} + C.$

RESULT 2 $\int \frac{1}{x}\,dx = \log|x| + C,$ *where* $x \neq 0.$

PROOF Clearly, either $x > 0$ or $x < 0$.

So, in this case, $\int \frac{1}{x}\,dx = \log|x| + C.$

Case I *When* $x > 0$

In this case, $|x| = x$.

$\therefore \quad \frac{d}{dx}[\log|x|] = \frac{d}{dx}(\log x) = \frac{1}{x}.$

So, we have, $\int \frac{1}{x}\,dx = \log|x| + C.$

Case II *When* $x < 0$

In this case $|x| = -x$.

$\therefore \quad \frac{d}{dx}[\log|x|] = \frac{d}{dx}[\log(-x)] = \frac{1}{(-x)}\cdot(-1) = \frac{1}{x}.$

So, we have $\int \frac{1}{x}\,dx = \log|x| + C.$

Thus, from both the cases, we have $\int \frac{1}{x}\,dx = \log|x| + C.$

FORMULAE

On the basis of differentiation and the definition of integration, we have the following results.

1. $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,\ n \neq -1 \Rightarrow \int x^n\,dx = \frac{x^{n+1}}{(n+1)} + C$
2. $\frac{d}{dx}(\log|x|) = \frac{1}{x} \Rightarrow \int \frac{1}{x}\,dx = \log|x| + C$
3. $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x\,dx = e^x + C$
4. $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x \Rightarrow \int a^x\,dx = \frac{a^x}{\log a} + C$
5. $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x\,dx = \sin x + C$

6. $\frac{d}{dx}(-\cos x) = \sin x \Rightarrow \int \sin x \; dx = -\cos x + C$

7. $\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x \, dx = \tan x + C$

8. $\frac{d}{dx}(-\cot x) = \text{cosec}^2 x \Rightarrow \int \text{cosec}^2 x \, dx = -\cot x + C$

9. $\frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x \, dx = \sec x + C$

10. $\frac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x \Rightarrow \int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x + C$

11. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

12. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{(1+x^2)} \Rightarrow \int \frac{1}{(1+x^2)} dx = \tan^{-1} x + C$

13. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \Rightarrow \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

With the help of the above formulae, it is easy to evaluate the following integrals.

EXAMPLE 1 *Evaluate:*

(i) $\int x^9 \, dx$ (ii) $\int \sqrt[3]{x} \, dx$ (iii) $\int dx$

(iv) $\int \frac{1}{x^2} dx$ (v) $\int \frac{1}{x^{1/3}} dx$ (vi) $\int 5^x dx$

SOLUTION Using the standard formulae, we have

(i) $\int x^9 dx = \frac{x^{(9+1)}}{(9+1)} + C = \frac{x^{10}}{10} + C.$

(ii) $\int \sqrt[3]{x} \, dx = \int x^{1/3} dx = \frac{x^{\left(\frac{1}{3}+1\right)}}{\left(\frac{1}{3}+1\right)} + C = \frac{3}{4} x^{4/3} + C.$

(iii) $\int dx = \int x^0 dx = \frac{x^{(0+1)}}{(0+1)} + C = x + C.$

(iv) $\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{(-2+1)}}{(-2+1)} + C = -\frac{1}{x} + C.$

(v) $\int \frac{1}{x^{1/3}} dx = \int x^{-1/3} dx = \frac{x^{\left(-\frac{1}{3}+1\right)}}{\left(-\frac{1}{3}+1\right)} + C = \frac{3}{2} x^{2/3} + C.$

(vi) $\int 5^x dx = \frac{5^x}{\log 5} + C.$

Some Standard Results on Integration

THEOREM 1 $\frac{d}{dx}\left\{\int f(x)\,dx\right\} = f(x).$

PROOF Let $\int f(x)\,dx = F(x).$ ... (i)

Then, $\frac{d}{dx}\{F(x)\} = f(x)$ [by def. of integral].

$\therefore \quad \frac{d}{dx}\left\{\int f(x)\,dx\right\} = f(x)$ [using (i)].

THEOREM 2 $\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx,$ *where k is a constant.*

PROOF Let $\int f(x)\,dx = F(x).$ ... (i)

Then, $\frac{d}{dx}\{F(x)\} = f(x).$ ... (ii)

$\therefore \quad \frac{d}{dx}\{k \cdot F(x)\} = k \cdot \frac{d}{dx}\{F(x)\} = k \cdot f(x)$ [using (ii)].

So, by the definition of an integral, we have

$\int \{k \cdot f(x)\}\,dx = k \cdot F(x) = k \cdot \int f(x)\,dx$ [using (i)].

EXAMPLE 2 *Evaluate:*

(i) $\int 3x^2 dx$ *(ii)* $\int 2^{(x+3)}\,dx.$

SOLUTION (i) $\int 3x^2 dx = 3\int x^2 dx = 3 \cdot \frac{x^3}{3} + C = x^3 + C.$

(ii) $\int 2^{(x+3)} dx = \int 2^x \cdot 2^3 dx = 8\int 2^x dx = 8 \cdot \frac{2^x}{\log 2} + C = \frac{2^{(x+3)}}{\log 2} + C.$

THEOREM 3 *(i)* $\int \{f_1(x) + f_2(x)\}\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx$

(ii) $\int \{f_1(x) - f_2(x)\}\,dx = \int f_1(x)\,dx - \int f_2(x)\,dx$

PROOF (i) Let $\int f_1(x)\,dx = F_1(x)$ and $\int f_2(x)\,dx = F_2(x).$ ... (i)

Then, $\frac{d}{dx}\{F_1(x)\} = f_1(x)$ and $\frac{d}{dx}\{F_2(x)\} = f_2(x).$ ... (ii)

Now, $\frac{d}{dx}\{F_1(x) + F_2(x)\} = \frac{d}{dx}\{F_1(x)\} + \frac{d}{dx}\{F_2(x)\}$

$= f_1(x) + f_2(x)$ [using (ii)].