Gauss - Jacobi interation method

This method is applicable to the system of eq^n in which leading diagonal elements of co-efficient matrix are dominant (large in magnitude) in their respective rows

Working Rule: consider the system of eq^{n}s

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{21}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Note Diagonal dominance property must be satisfied

i.e
$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{31}| > |a_{32}| + |a_{32}|$$

Re writing the eq^{n}s for x, y, z resp.

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z)$$
$$y = \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z)$$
$$z = \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y)$$

Iteration 1: put $x = x_0$, $y = y_0$, $z = z_0$

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}y_0 - a_{13}z_0)$$
$$y_1 = \frac{1}{a_{22}}(b_2 - a_{21}x_0 - a_{23}z_0)$$
$$z_1 = \frac{1}{a_{33}}(b_3 - a_{31}x_0 - a_{32}y_0)$$

Again, substituting these values of x, y, z, the next approximation is obtained.

The above iteration process is continued until two successive approximations are equal.

Ex Solve the following system of eqns by Gauss - Jacobi method.

$$6x+2y-z=4$$
$$x+5y+z=3$$
$$2x+y+4z=27$$

Soln Rewriting the eqns

$$\left.\begin{aligned} x &= \frac{1}{6}(4-2y+z) \\ y &= \frac{1}{5}(3-x-z) \\ z &= \frac{1}{4}(27-2x-y) \end{aligned}\right\} \text{———} \ (1)$$

Iteration 1: put $x=x_0=0$, $y=y_0=0$, $z=z_0=0$ in eqn (1)

$$x_1 = \frac{1}{6}(4-2y_0+z_0) = 0.6667$$
$$y_1 = \frac{1}{5}(3-x_0-z_0) = 0.6$$
$$z_1 = \frac{1}{4}(27-2x_0-y_0) = 6.75$$

Iteration 2: put x_1, y_1, z_1 in eqn (1)

$$x_2 = \frac{1}{6}(4-2y_1+z_1) \ \cancel{=1.5917}$$
$$= \frac{1}{6}(4-2\times 0.6+6.75) = 1.5917$$

$y_2 = \frac{1}{5}(3 - x_1 - z_1)$

$= \frac{1}{5}(3 - 0.6667 - 6.75) = -0.8833$

$z_2 = \frac{1}{4}(27 - 3x_1 - y_1)$

$= \frac{1}{4}(27 - 3 \times 0.6667 - 0.6) = 6.2666.$

Iteration 3: put x_2, y_2, z_2 in eqn ①

$x_3 = \frac{1}{6}(4 - 2y_2 + z_2)$

$= \frac{1}{6}[4 - 2 \times (-0.8833) + 6.2666] = 2.0055$

$y_3 = \frac{1}{5}(3 - x_2 - z_2) = \cancel{0.9717}\ \frac{1}{5}(3 - 1.5917 - 6.2666)$

$= 0.9717$

$z_3 = \frac{1}{4}(27 - 2x_2 - y_2)$

$= \frac{1}{4}[27 - 2 \times 1.5917 - (-0.8833)] = 6.1750$

Iteration 4: put $x = x_3, y = y_3, z = z_3$ in eqn ①

$x_4 = \frac{1}{6}(4 - 2y_3 + z_3)$

$= \frac{1}{6}(4 - 2 \times 0.9717 + 6.1750) = 2.0197$

$y_4 = \frac{1}{5}(3 - x_3 - z_3) = \frac{1}{5}(3 - 2.0055 - 6.1750)$

$\cancel{= 2.0197} = -1.0361$

$z_4 = \frac{1}{4}(27 - 2x_3 - y_3) = \frac{1}{4}(27 - 2 \times 2.0055 - 0.9717)$

$= 5.9902$

$y_2 = \frac{1}{5}(3 - x_1 - z_1)$

$= \frac{1}{5}(3 - 0.6667 - 6.75) = -0.8833$

$z_2 = \frac{1}{4}(27 - 3x_1 - y_1)$

$= \frac{1}{4}(27 - 3\times 0.6667 - 0.6) = 6.2666.$

Iteration 3: put x_2, y_2, z_2 in eqn ①

$x_3 = \frac{1}{6}(4 - 2y_2 + z_2)$

$= \frac{1}{6}[4 - 2\times(-0.8833) + 6.2666] = 2.0055$

$y_3 = \frac{1}{5}(3 - x_2 - z_2) = \cancel{0.9717}\ \frac{1}{5}(3 - 1.5917 - 6.2666)$

$= 0.9717$

$z_3 = \frac{1}{4}(27 - 2x_2 - y_2)$

$= \frac{1}{4}[27 - 2\times 1.5917 - (-0.8833)] = 6.1750$

Iteration 4: put $x = x_3, y = y_3, z = z_3$ in eqn ①

$x_4 = \frac{1}{6}(4 - 2y_3 + z_3)$

$= \frac{1}{6}(4 - 2\times 0.9717 + 6.1750) = 2.0197$

$y_4 = \frac{1}{5}(3 - x_3 - z_3) = \frac{1}{5}(3 - 2.0055 - 6.1750)$

$= \cancel{2.0197} = -1.0361$

$z_4 = \frac{1}{4}(27 - 2x_3 - y_3) = \frac{1}{4}(27 - 2\times 2.0055 - 0.9717)$

$= 5.9902$

Iteration 5: put $x = x_4,\ y = y_4,\ z = z_4$ in eqn ①

$x_5 = \frac{1}{6}(4 - 2y_4 + z_4)$

$= \frac{1}{6}[4 - 2\times(-1.0361) + 5.9902] = \cancel{2.0404}\ 2.0104$

$y_5 = \frac{1}{5}(3 - x_4 - z_4) = \cancel{\frac{1}{5}(3 - (-1.0361) -}$

$= \frac{1}{5}(3 - 2.0197 - 5.9902) = -1.0020$

$z_5 = \frac{1}{4}(27 - 2x_4 - y_4) = \frac{1}{4}[27 - 2.0197\times 2 - (-1.0361)]$

$= 5.9992.$

Iteration 6: put $x = x_5,\ y = y_5,\ z = z_5$ in eqn ①

$x_6 = \frac{1}{6}(4 - 2\times y_5 + z_5) = \frac{1}{6}(4 - 2\times(-1.0020) + 5.9992)$

$= 2.0005 \approx 2.00$

$y_6 = \frac{1}{5}(3 - x_5 - z_5) = \frac{1}{5}(3 - 2.0104 - 5.9992)$

$= -1.0019 \approx -1.00$

$z_6 = \frac{1}{4}(27 - 2x_5 - y_5) = \frac{1}{4}[27 - 2\times 2.0104 - (1.0020)]$

$= 5.9953 \approx 6.00$

Iteration 7: put $x = x_6,\ y = y_6,\ z = z_6$ in eqn ①

$x_7 = \frac{1}{6}(4 - 2y_6 + z_6) = \frac{1}{6}(4 - 2\times(-1.0019) + 5.9953)$

$= 1.9998 \approx 2.00$

$y_7 = \frac{1}{5}(3 - x_6 - z_6) = \frac{1}{5}(3 - 2.0005 - 5.9953)$

$= -0.9992 \approx -1.00$

$$z_7 = \frac{1}{4}(27 - 2x_6 - y_6)$$

$$= \frac{1}{4}[27 - 2 \times 2.0005 - (-1.0019)]$$

$$= 6.0002 \approx 6.00$$

Since the 6th and 7th iterations values are same (upto 2 decimal places)

Hence the approximate solⁿ is

$$x = 2, \quad y = -1, \quad z = 6.$$

H.W Solve,

$$10x - 5y - 2z = 3$$
$$4x - 10y + 3z = -3$$
$$x + 6y + 10z = -3$$

Ans $x = 0.341$
$y = 0.285$
$z = -0.505$