Ex. When simple harmonic waves of length λ are propagated over the surface of deep water, prove that at a point whose depth below the undisturbed surface is h, the pressure at the instants when the disturbed depth at the point is $h+\eta$ bears to the undisturbed pressure at the same point in the ratio

$$1+\left(\frac{\eta}{h}\right)e^{-\frac{2\pi h}{\lambda}} : 1 .$$

when atmospheric pressure and surface tension are neglected.

Soln.

The complex potential for waves in deep water is

$$w = ac\, e^{-i(mz-nt)} \longrightarrow (i)$$

$$= ac\, e^{-i\{m(x+iy)-nt\}}$$

$$= ac\, e^{my} e^{-i(mx-nt)}$$

$$= ac\, e^{my}\left\{\cos(mx-nt) - i\sin(mx-nt)\right\}$$

$= \phi + i\psi$, where ϕ is the velocity potential and ψ is the stream function.

Thus $\phi = ac\, e^{my}\cos(mx-nt) \longrightarrow$ (ii)

Now, the wave velocity c is given by

$$c = \frac{n}{m}$$

$$\Rightarrow c^2 = \frac{n^2}{m^2} = \frac{g}{m}$$

Now, $c = \frac{c^2}{c} = \frac{\frac{g}{m}}{\frac{n}{m}} = \frac{g}{n} \longrightarrow$ (iii)

from (ii) and (iii), we have,

$$\phi = \frac{ag}{n} e^{my}\cos(mx-nt)$$

when velocity is very small then the pressure eqn.

may be taken as

$$\frac{p}{\rho} + gy - \frac{\partial \phi}{\partial t} = A \text{ (constant)} \longrightarrow \text{(iv)}$$

Initially, when $y=0$, pressure (atmospheric) is neglected so that $A=0$ and hence (iv) gives,

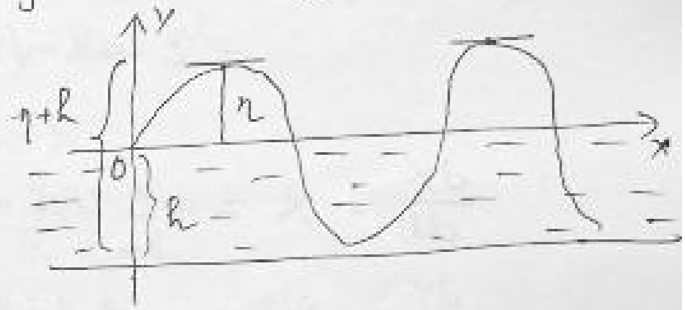
$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - gy = age^{my}\sin(mx-nt) - gy$$

$$= g\left(e^{my}\eta - y\right) \quad ; \quad \eta = a\sin(mx-nt)$$

Let p_1 be the pressure when $y=-h$.

i.e. p_1 be the pressure at a depth h while the height of the point from the free surface is $h+\eta$.

i.e. we have



$$\frac{p_1}{\rho} = g\left(e^{-mh}\eta + h\right)$$

$$= gh\left(1 + \frac{\eta}{h}e^{-mh}\right) \longrightarrow \text{(v)}$$

Also, let the pressure is p_2 when the undisturbed depth is h, In this case,

$$p_2 = \rho g h \longrightarrow \text{(vi)}$$

from (v) and (vi),

$$\frac{p_1}{p_2} = \frac{gh\left(1 + \frac{\eta}{h}e^{-mh}\right)}{gh}$$

$$\Rightarrow p_1 : p_2 = 1 + \frac{\eta}{h}e^{-mh} : 1$$

#

Ex. Show that particles on waves in a canal of small depth describe ellipses about their position of equilibrium, whereas they are circles for sea waves.

Soln. Let (x, y) be the relative position of a particle w.r.to its mean position (x, y). Then neglecting the difference of velocity at (x, y) and $(x+X, y+Y)$, we get,

$$\frac{dX}{dt} = -\frac{\partial\phi}{\partial x}$$

$$\frac{dY}{dt} = -\frac{\partial\phi}{\partial y}$$

where $\phi = \frac{ac}{\sinh mh} \cdot \cosh m(y+h) \cos(mx - nt)$

$= c \cosh m(y+h) \cos(mx-nt)$, if $c = \frac{ac}{\sinh mh}$

$$\therefore \frac{dX}{dt} = \dot{X} = mC \cosh m(y+h) \sin(mx-nt) \longrightarrow (i)$$

$$\frac{dY}{dt} = \dot{Y} = -mC \sinh m(y+h) \cos(mx-nt) \longrightarrow (ii)$$

Integrating (i) and (ii) w.r.to 't', we have,

$$X = \frac{mC}{n} \cosh m(y+h) \cos(mx-nt)$$

$$\Rightarrow X = A\cos(mx-nt) \longrightarrow (iii)$$

$$Y = \frac{mC}{n} \sinh m(y+h) \sin(mx-nt)$$

$$= B\sin(mx-nt) \longrightarrow (iv)$$

~~Integrating (i) and (ii) w.r. to 't', we have,~~

$$\cancel{X = \frac{mC}{n} \cosh m(y+h) \cos(mx-nt)}$$

(ϕ is adjusted in such a way that the constants of integration are neglected)

Squaring and adding (iii) and (iv),

$$\frac{X^2}{A^2}+\frac{Y^2}{B^2}=\cos^2(mx-nt)+\sin^2(mx-nt)$$
$$=1.$$

This shows that the particles describe ellipse about their mean position. If E be the eccentricity of the ellipse, then

$$B^2=A^2(1-E^2)=A^2-A^2E^2$$

or, $A^2E^2=A^2-B^2$

or, $AE=\sqrt{A^2-B^2}=\frac{mc}{n}$

or, $\cosh^2 m(y+h)-\sinh^2 m(y+h)=1$

or, $AE=\frac{m}{a}=\frac{ac}{\sinh mh}=\frac{n}{\sinh mh}$, as $c=\frac{n}{m}$

and distance between foci $=2AE$

$$=\frac{2n}{\sinh mh}$$

$$=\text{constant}.$$

Also, the length of the minor axis $B=0$ when $y=-h$.
Thus fluid particles describe ellipse about their mean position, the distance between the foci $2a\,\mathrm{cosech}\,mh$ which is the same for all such ellipses. The major and minor both axes decrease as depth increases. At the bottom, the ellipse degenerates into a straight line, here the particle have to and fro motion.

In case of deep water,

$$\dot{X}=-\frac{\partial\phi}{\partial x}=-\frac{\partial}{\partial x}\left[ace^{my}\cos(mx-nt)\right]=mace^{my}\sin(mx-nt)\rightarrow\text{(v)}$$

$$\dot{Y}=-\frac{\partial\phi}{\partial y}=-\frac{\partial}{\partial y}\left[ace^{my}\cos(mx-nt)\right]=-mace^{my}\cos(mx-nt)\longrightarrow\text{(vi)}$$

Integrating (v) and (vi) w.r. to 't', we have,

$$X = -\frac{mac}{(-n)} e^{my} \cos(mx - nt) \longrightarrow (vii)$$

$$= ac\, e^{my} \cos(mx - nt) \quad ; \because c = \frac{n}{m}$$

$$Y = \frac{-acm}{(-m)} e^{my} \sin(mx - nt)$$

$$= ac\, e^{my} \sin(mx - nt) \longrightarrow (viii)$$

Squaring and adding (vii) and (viii),

$$X^2 + Y^2 = (ac\, e^{my})^2 \left[\cos^2(mx - nt) + \sin^2(mx - nt)\right]$$

$$= (ac\, e^{my})^2$$

This is the equation of a circle.

Proved.

Standing or stationary waves:

Two simple harmonic progressive waves of same amplitude, same wave length and period, travelling in opposite directions are given by

$$y_1 = \frac{1}{2} a \sin(mx - nt), \quad y_2 = \frac{1}{2} a \sin(mx + nt)$$

By, the principle of superposition, the resultant of these two waves is given by

$$y = y_1 + y_2$$

$$= \frac{1}{2} a \sin(mx - nt) + \frac{1}{2} a \sin(mx + nt)$$

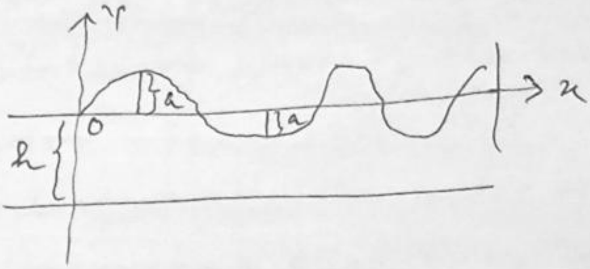
$$= a \sin mx \cos nt \quad ; \quad y = a \sin(mx - nt) \text{ at } t = 0$$

$$= (a \cos nt) \sin mx \quad \quad y = a \sin mx.$$

$$= A(t) \sin mx$$

A motion of this type is called stationary motion.

Now, $y = A(t)\sin mx$ does not represent a moving curve are progressive wave. At any position, surface of the wave goes up and down due to the variation of the amplitude with time. But any instant, it is a fixed sine curve with amplitude a constant, which therefore varies between 0 to a. Thus a wave of this type is not propagate.
The points of intersection of the curve with x-axis are given by



$$\sin mx = 0$$

$$\Rightarrow \sin mx = \sin n\pi$$

$$\Rightarrow mx = n\pi$$

$$\Rightarrow x = \frac{n\pi}{m} = \frac{\lambda n}{2} \quad ; \because \lambda = \frac{2\pi}{m}, \text{ for } n = 1, 2, 3, \ldots$$

$$\therefore x = \frac{\lambda}{2}, \lambda, 3\lambda/2, \ldots$$

These points are known as nodes and the intermediate points whose the amplitude is maximum are called antinodes.

Again, if $y_3 = a \sin mx \cos nt$

$y_4 = a \cos mx \cdot \sin nt$

be two standing waves, the result of superposing these is the elevation

$$y = y_3 \pm y_4$$

i.e. $y = a \sin(mx \pm nt)$

Hence the progressive waves can be regarded as the combination of two systems of stationery waves of the same amplitude, wave length and period, the crest, and throughs of one system coinciding with the nodes of the other and their phases differs by a quarter period.

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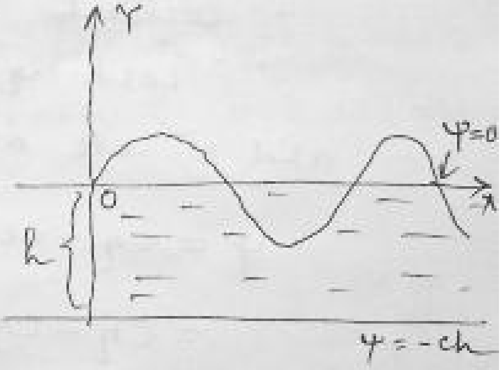
Q: Distinguish between progressive waves and stationary waves.

§: Progressive waves reduced to steady motion:

The complex potential of a simple sine wave $\eta = a \sin(mx - nt)$, when the origin lies in the undisturbed level is

$$W_0 = \frac{ac}{\sinh mh} \cdot \cos\left[m(z + ih) - nt\right] \rightarrow (1)$$



Let us impose on the whole mass a velocity equal and opposite to the velocity of propagation to the waves, the axes and the wave profile possessing the same relative velocity gets fixed in space and the problem reduces to one of steady motion. The fluid begins to flows with velocity c in the negative direction of x-axis. Contributing the term cz in the complex potential.

Hence the final form of complex potential giving the steady motion value is

$$w = cz + \frac{ac}{\sinh mh} \cos[m(z+ih)] \longrightarrow (11)$$

It only remains to show that the free surface and the bottom of the liquid satisfy the conditions for stream lines.

From (11), equating the real and imaginary parts,

$$\phi = cx + \frac{ac}{\sinh mh} \cos mx \cosh m(y+h)$$

$$\psi = cy - \frac{ac}{\sinh mh} \sin mx \sinh m(y+h)$$

At the bottom $y = -h$, so that $\boxed{\psi = -ch}$

For the free surface $y = \eta \; [= a \sin mx]$

$$\therefore \psi = c\eta - \frac{ac}{\sinh mh} \sin mx \quad ;\Big) \eta = a \sin(mx - nt)$$

$$\times \sinh m(y+h$$

$$= c\eta - \frac{ac}{\sinh mh} \sin mx \left[\sinh mh \cosh m\eta + \cosh mh \sinh m\eta \right]$$

~~$\because \cosh mh$~~

$\because \cosh m\eta = 1, \; \sinh m\eta = m\eta$ (approximately)

and η^2 etc. are negligible, we get,

$$\psi = c\eta - ac \sin mx - acm\eta \sin mx \coth mh$$

$$= c\eta - c\eta - cm\eta^2 \coth mh$$

$$\approx 0$$

Thus the free surface and the bottom surface are stream lines.

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