$y_1 = 3,\ y_4 = 24.$

4.4 NEWTON'S FORWARD INTERPOLATION FORMULA

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the $(n + 1)$ values $x_0, x_1, x_2, \ldots, x_n$ of the independent variable x. Let the values x be equally spaced, i.e.,

$$x_r = x_0 + rh,\ r = 0, 1, 2, \ldots, h$$

where h is the interval of differencing. Let $\phi(x)$ be a polynomial of the nth degree in x taking the same values as y corresponding to $x = x_0, x_1, \ldots, x_n$, then, $\phi(x)$ represents the continuous function $y = f(x)$ such that $f(x_r) = \phi(x_r)$ for $r = 0, 1, 2, \ldots, n$ and at all other points $f(x) = \phi(x) + R(x)$ where $R(x)$ is called the *error term* (Remainder term) of the interpolation formula. Ignoring the error term let us assume

$$f(x) \approx \phi(x) \approx a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_o)(x - x_1)\ldots(x - x_{n-1}) \quad (3)$$

the constants $a_0, a_1, a_2, \ldots, a_n$ can be determine as follows.

Putting $x = x_0$ in (3) we get

$$f(x_0) \approx \phi(x_0) = a_0$$

$$\Rightarrow y_0 = a_0$$

putting $x = x_1$ in (3) we get

$$f(x_1) \approx \phi(x_1) = a_0 + a_1(x_1 - x_0) = y_0 + a_1 h$$

$$\therefore\ y_1 = y_0 + a_1 h$$

$$\Rightarrow a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}.$$

Putting $x = x_2$ in (3) we get

$$f(x_2) \approx \phi(x_2) = a_0 + a_1(x_2 - x_0)\, a_2(x_2 - x_0)(x_2 - x_1)$$

$$\therefore\ y_2 = y_0 + \frac{\Delta y_0}{h}(2h) + a_2(2h)(h)$$

$$\Rightarrow y_2 = y_0 + 2(y_1 - y_0) + a_2(2h^2)$$

$$\Rightarrow a_2 = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly by putting $x = x_3, x = x_4 \ldots,\ x = x_n$ in (3) we get

$$a_3 = \frac{\Delta^3 y_0}{3!h^3},\ a_4 = \frac{\Delta^4 y_0}{4!h^4}, \ldots, a_n = \frac{\Delta^n y_0}{n!h^n}$$

putting the values of $a_0, a_1, \ldots, a_n$ in (3) we get

$$f(x) \approx \phi(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) +$$

$$\frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \ldots +$$

$$\frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)(x - x_{n-1}) \quad (4)$$

Writing $u = \dfrac{x - x_0}{h}$, we get $x - x_0 = uh$

$$x - x_1 = x - x_0 + x_0 - x_1$$

$$= (x - x_0) - (x_1 - x_0) = uh - h = (u - 1)h$$

Writing $u = \frac{x - x_0}{h}$, we get $x - x_0 = uh$

$$x - x_1 = x - x_0 + x_0 - x_1$$

$$= (x - x_0) - (x_1 - x_0) = uh - h = (u - 1)h$$

Similarly

$$x - x_2 = (u - 2)h$$

$$x - x_3 = (u - 3)h$$

$$\cdots$$

$$x - x_{n-1} = (u - n + 1)h$$

Equation (4) can be written as

$$\phi(x) = y_0 + u\frac{\Delta y_0}{1!} + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_o.$$

The above formula is called *Newton's forward interpolation formula.*

Note:

1. Newton forward interpolation formula is used to interpolate the values of y near the beginning of a set of tabular values.
2. y_0 may be taken as any point of the table, but the formula contains only those values of y which come after the value chosen as y_0.

Example 4.4 *Given that*

$\sqrt{12500} = 111.8034$, $\sqrt{12510} = 111.8481$

$\sqrt{12520} = 111.8928$, $\sqrt{12530} = 111.9375$

find the value of $\sqrt{12516}$.

Solution The difference table is

x	$y = \sqrt{x}$	Δy	$\Delta^2 y$
12500 x_0	111.8034 y_0		
		0.0447 Δy_0	
12510	111.8481		0 $\Delta^2 y_0$
		0.0447	
12520	111.8928		0
		0.0447	
12530	111.9375		

We have $x_0 = 12500$, $h = 10$ and $x = 12516$

$$u = \frac{x - x_0}{h} = \frac{12516 - 12510}{10} = 1.6$$

from Newton's forward interpolation formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots$$

$$\Rightarrow f(12516) = 111.8034 + 1.6 \times 0.0447 + 0 + \ldots$$

$$= 111.8034 + 0.07152 = 111.87492$$

$$\therefore \sqrt{12516} = 111.87492.$$

Example 4.5 *Evaluate $y = e^{2x}$ for $x = 0.05$ using the following table*

x	0.00	0.10	0.20	0.30	0.40
$y = e^{2x}$	1.000	1.2214	1.4918	1.8221	2.255

Solution The difference table is

x	$y = e^{2x}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.000	1.0000				
		0.2214			
0.10	1.2214		0.0490		
		0.2704		0.0109	
0.20	1.4918		0.0599		0.0023
		0.3303		0.0132	
0.30	1.8221		0.0731		
		0.4034			
0.40	2.2255				

We have $x_0 = 0.00$, $x = 0.05$, $h = 0.1$.

$$\therefore u = \frac{x - x_0}{h} = \frac{0.05 - 0.00}{0.1} = 0.5$$

Using Newton's forward formula

$$f(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots$$

$$f(0.05) = 1.0000 + 0.5 \times 0.2214 + \frac{0.5(0.5-1)}{2}(0.0490) + \frac{0.5(0.5-1)(0.5-2)}{6}(0.0109) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(0.0023)$$

$$= 1.000 + 0.1107 - 0.006125 + 0.000681 - 0.000090 = 1.105166$$

$$\therefore f(0.05) \approx 1.052.$$

Example 4.6 *The values of sin x are given below for different values of x. Find the value of sin 32°*

x	30°	35°	40°	45°	50°
$y = \sin x$	0.5000	0.5736	0.6428	0.7071	0.7660

Solution $x = 32°$ is very near to the starting value $x_0 = 30°$. We compute sin 32° by using Newton's forward interpolation formula.

$\therefore\ f(0.05) = 1.052.$

Example 4.6 *The values of sin x are given below for different values of x. Find the value of sin 32°*

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Solution $x = 32°$ is very near to the starting value $x_0 = 30°$. We compute sin 32° by using Newton's forward interpolation formula.

The difference table is

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30°	0.5000				
		0.0736			
35°	0.5736		–0.0044		
		0.0692		–0.005	
40°	0.6428		–0.0049		0
		0.0643		–0.005	
45°	0.7071		–0.0054		
		0.0589			
50°	0.7660				

$$u = \frac{x - x_0}{h} = \frac{32° - 30°}{5} = 0.4.$$

We have $y_0 = 0.5000,\ \Delta y_0 = 0.0736,\ \Delta^2 y_0 = -0.0044,\ \Delta^3 y_0 = -0.005$

putting these values in Newton's forward interpolation formula we get

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$\Rightarrow f(32°) = 0.5000 + 0.4 \times 0.0736 + \frac{(0.4)\,(0.4-1)}{2}(-0.0044) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(-0.0005)$$

$$f(32°) = 0.5000 + 0.02944 + 0.000528 - 0.00032 = 0.529936 = 0.299.$$

Example 4.7 *In an examination the number of candidates who obtained marks between certain limits were as follows:*

Marks	30–40	40–50	50–60	60–70	70–80
No. of Students	31	42	51	35	31

Find the number of candidates whose scores lie between 45 and 50.

Solution Fist of all we construct a cumulative frequency table for the given data.

Upper limits of the class intervals	40	50	60	70	80
Cumulative frequency	31	73	124	159	190

The difference table is

x *marks*	y *cumulative frequencies*	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		–25	

$$\Rightarrow f(32^\circ) = 0.5000 + 0.4 \times 0.0736 + \frac{(0.4)(0.4-1)}{2}(-0.0044) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(-0.0005)$$

$$f(32^\circ) = 0.5000 + 0.02944 + 0.000528 - 0.00032 = 0.529936 = 0.299.$$

Example 4.7 *In an examination the number of candidates who obtained marks between certain limits were as follows:*

Marks	*30–40*	*40–50*	*50–60*	*60–70*	*70–80*
No. of Students	*31*	*42*	*51*	*35*	*31*

Find the number of candidates whose scores lie between 45 and 50.

Solution Fist of all we construct a cumulative frequency table for the given data.

Upper limits of the class intervals	40	50	60	70	80
Cumulative frequency	31	73	124	159	190

The difference table is

x *marks*	y *cumulative frequencies*	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		–25	
60	124		–16		37
		35		12	
70	159		–4		
		31			
80	190				

we have

$$x_0 = 40,\ x = 45,\ h = 10$$

$$u = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

and

$$y_0 = 73,\ \Delta y_0 = 42,\ \Delta^2 y_0 = 9,\ \Delta^3 y_0 = -25,\ \Delta^4 y_0 = 37.$$

From Newton's forward interpolation formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^2 y_0 + \ldots$$

$$\therefore f(45) = 31 + (0.5)(42) + \frac{(0.5)(-0.5)}{2} \times 9 + \frac{(0.5)(0.5-1)(0.5-2)}{6}(-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24} \times (37)$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4452 = 47.8673$$

$$= 48 \text{ (approximately)}$$

$\therefore$ The number of students who obtained marks less than 45 = 48, and the number of students who scored marks between 45 and 50 = 73 – 48 = 25.