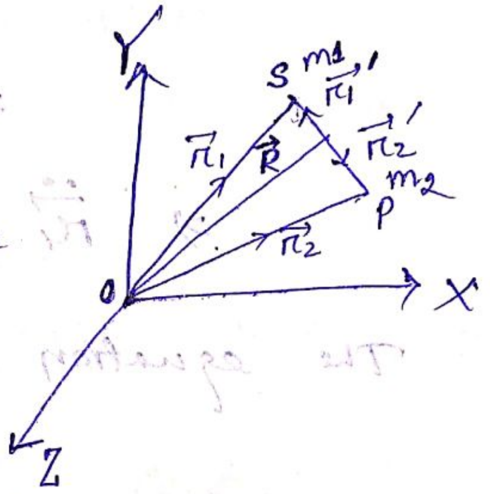
... moved uniformly in a straight line in space.

Relative Motion:



2014 Let us take the origin at the centre of mass of the two body systems.

Let $\vec{r_1}$ and $\vec{r_2}$ be the position vectors of m_1, m_2 with respect to the centre of mass C of the system.

Let us suppose that the position vector of C with respect to O is $\vec{R}$ and those of m_1 and m_2 are $\vec{r_1}$ and $\vec{r_2}$ respectively.

Also let $\vec{r}$ is the position vector of m_2 relative to m_1, then

$$\vec{r_1} = \vec{R} + \vec{r_1}' \; ; \; \vec{r_2} = \vec{R} + \vec{r_2}'$$

$$\vec{r} = \vec{r_2}' - \vec{r_1}' \longrightarrow (1)$$

from (1), $m_1\vec{r_1} = m_1\vec{R} + m_1\vec{r_1}'$

$$\Rightarrow m_1\ddot{\vec{r_1}} = m_1\ddot{\vec{r_1}}' \quad [\because \ddot{\vec{R}} = 0]$$

Similarly, $m_2\ddot{\vec{r_2}} = m_2\ddot{\vec{r_2}}'$

$$m_1\ddot{\vec{r_1}}' = \frac{G m_1 m_2}{r^3}(\vec{r_2}' - \vec{r_1}') \rightarrow (2)$$

and $$m_2\ddot{\vec{r_2}}' = -\frac{G m_1 m_2}{r^3}(\vec{r_2}' - \vec{r_1}') \rightarrow (3)$$

Since the origin is taken at the centre of mass therefore we get $m_1\vec{r_1}' + m_2\vec{r_2}' = 0$

$$\Rightarrow \frac{m_1}{m_2} = -\frac{\vec{r_2}'}{\vec{r_1}'} \rightarrow (4)$$

using (4), eliminating $\vec{r_2}'$ from (2), we get

$$m_1\ddot{\vec{r_1}}' = G m_1 m_2\left(\frac{-\vec{r_1}' m_1}{m_2} - \vec{r_1}'\right)$$

$$= -G m_1 m_2\left(1 + \frac{m_1}{m_2}\right)\frac{\vec{r_1}'}{r^3} \rightarrow (5)$$

Similarly, eliminating $\vec{r_1}'$ from (3), we get

$$m_2\ddot{\vec{r_2}}' = -G m_1 m_2\left(1 + \frac{m_2}{m_1}\right)\frac{\vec{r_2}'}{r^3} \rightarrow (6)$$

Let $M = m_1 + m_2$ be the total mass of the pair of mass m_1 and m_2

then ⑤ and ⑥ can be written as

$$\ddot{\vec{r}}_1' = -\frac{GM}{r^3}\vec{r}_1' \longrightarrow ⑦$$

$$\ddot{\vec{r}}_2' = -\frac{GM}{r^3}\vec{r}_2' \longrightarrow ⑧$$

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Thus substituting eqⁿ ⑧ from eqⁿ ⑦, we get

$$\ddot{\vec{r}}_2' - \ddot{\vec{r}}_1' = -\frac{GM}{r^3}\left(\vec{r}_2' - \vec{r}_1'\right)$$

$$\Rightarrow \ddot{\vec{r}} = -\frac{GM}{r^3}\vec{r} \quad \text{using (i)}$$

$$= -\frac{G(m_1+m_2)\vec{r}}{r^3} \longrightarrow ⑨$$

This differential eqⁿ expresses the acceleration of m_2, relative to m_1. In the case of Planetary System, we may take m_1 as the ~~sum~~ Sun and m_2 as a Planet.

In the case of an artificial satellite, m_1 can be taken as the mass of Earth and m_2 that of the satellite.

In cartesian form, eqⁿ ⑨ is equivalent

to three-second order differential equations, requiring six constant of integration for their solution, if we know the position (three co-ordinates) and the velocity (three components) at any instant, these six constants of integration can be determined.

Since our affonts will be devoted to studying the motion of artificial satellites, ballistic missiles on space probes orbiting the Earth on the motion of a planet about the sun, the mass of the orbiting body is very mass less than that of the central body, Hence we may take

$$G(m_1+m_2) \approx G m_1 \quad (\because m_1 \gg m_2)$$

Let $GM = \mu$, then the eqn of motion (9) of the two body problem can be written as $\ddot{\vec{r}} = -\dfrac{\mu(\vec{r})}{r^3}$

$$= -\frac{\mu r \hat{r}}{r^3}$$

$$= -\frac{\mu \hat{r}}{r^2} \longrightarrow (10)$$

2014 This shows that the motion of the planet (or satellite) relative to the Sun follow the inverse square law

from (10), $\vec{r} \times \ddot{\vec{r}} = 0$

$$\Rightarrow \vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\vec{r} \times \vec{v}\right) = 0$$

On integration, we get $\vec{r} \times \vec{v} = \vec{h}$

$\vec{h}$ is a constant vector $\longrightarrow (11)$

From (11), we get that angular momentum of the system is constant.

Now, taking the dot product of eqn (11) with $\vec{r}$, we get $\vec{r} \cdot (\vec{r} \times \vec{v}) = \vec{r} \cdot \vec{h}$

$$\Rightarrow \vec{r} \cdot \vec{h} = 0 \longrightarrow (12) \quad , \left[\vec{r} \cdot (\vec{r} \times \vec{v}) = [\vec{r}, \vec{r}, \vec{v}] = 0 \right.$$

The eqn. (12) represents a plane through S perpendicular to the ~~$\vec{h} = h_1\hat{i} + h_2\hat{j} + h_3\hat{k}$~~ vector $\vec{h}$.

$\vec{h} = h_1\hat{i} + h_2\hat{j} + h_3\hat{k}$

Thus the orbit of the planet (or satellite) around the sun (or Earth) is a plane curve.