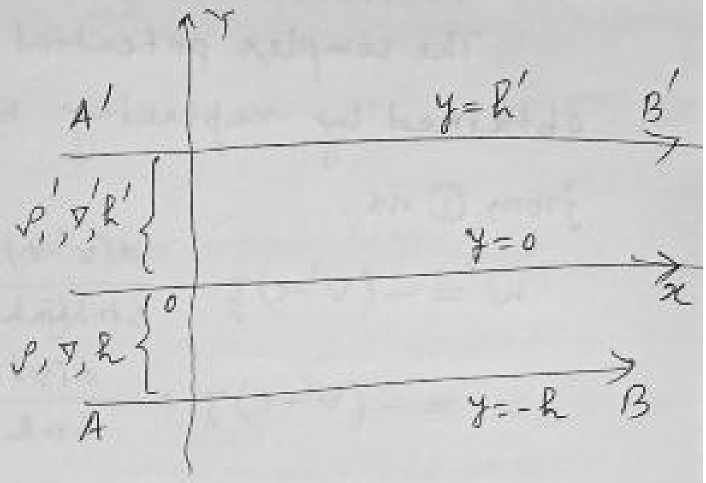
§: Waves at an interface of two Liquids:

Let a liquid of density ρ', depth h' move with velocity v' over another liquid of density ρ, depth h which moves in the same direction with velocity v. Let the liquid be bounded above and below by two fixed rigid horizontal planes $A'B'$ and AB respectively.



If the system is slightly disturbed, the waves may be propagated at the common surface (or interface) of two liquids.

Let C be the velocity of propagation of oscillatory waves at the interface of two liquids in the direction in which liquids are moving. Let us take x-axis in the geometrically interface and y-axis superimpose on the whole mass a velocity equal and opposite to the velocity of propagation of waves. Thus reducing the wave profile to rest in space and changing the velocities of the streams to $v'-c$ and $v-c$ as shown in figure.

For steady motion, the complex potential is given by

$$\omega = cz + \frac{ac}{\sinh mh} \cdot \cos m(z + ih)$$

The complex potential for the lower liquid moving with $-(V-c)$ in the $-ve$ direction of x-axis is

$$W = -(V-c)z + \frac{a(V-c)}{\sinh mh} \cdot \cos m(z + ih) \quad \rightarrow (1)$$

which can be verified by observing that the wave profile $\eta = a\sin mx$ gives $\psi = 0$ or vice-versa.

The complex potential for the upper liquid is similarly obtained by replacing w by w', h by $-h'$ and v by v' from (1) as,

$$w' = -(v'-c)z - \frac{a(v'-c)}{\sinh(-mh')} \cdot \cos m(z - ih')$$

$$= -(v'-c)z + \frac{a(v'-c)}{\sinh mh'} \cos m(z - ih) \longrightarrow (II)$$

For the speed in the lower liquid, we have

$$q^2 = \frac{dW}{dz}\,\frac{d\bar{W}}{dz}$$

$$\Rightarrow q^2 = \left[-(v-c) + \frac{a(v-c)}{\sinh mh}\, m \sin m(z+ih)\right]$$

$$\times \left[-(v-c) + \frac{a(v-c)}{\sinh mh} \cdot m \sin m(z-ih)\right]$$

$$= (v-c)^2 - \frac{2ma(v-a)^2}{\sinh mh} \cdot \sin mx \cosh m(y+h)$$

; (neglecting the term containing a^2)

$$= (v-c)^2\left[-1 - \frac{2ma}{\sinh mh} \cdot \sin mx \cosh m(y+h)\right] \rightarrow (III)$$

To obtain the speed q_0 at the interface, we put $q = q_0$ and $y = 0$ in (III)

$$q^2 = (v-c)^2\left[1 - 2ma \coth mh \sin mx\right]$$

$$= (v-c)^2\left[1 - 2m\eta \coth mh\right] \quad \text{for } \eta = a\sin mx$$

To obtain the speed q_0 at the interface due to upper liquid, replace h by $-h'$, v by v', q by q_0' in (III), we have,

$$q_0^2 = (V'-c)^2\left[1+2m\eta \coth mh'\right]$$

The expression for the pressure at the interface are

$$\frac{p'}{\rho'} + \frac{1}{2} q_0'^2 + g\eta = \text{constant}$$

$$\Rightarrow p' + \frac{1}{2}\rho' q_0'^2 + \rho' g\eta = \text{constant} \text{ for upper Liquid.}$$

and $$\frac{p}{\rho} + \frac{1}{2} q_0^2 + g\eta = \text{constant}$$

$$\Rightarrow p + \frac{1}{2}\rho q_0^2 + \rho g\eta = \text{constant} \text{ for lower liquid.}$$

Now, subtracting,

$$\frac{1}{2}\rho' q_0'^2 - \frac{1}{2}\rho q_0^2 + g\eta(\rho'-\rho) = 0$$, (as the pressure is to be continuous across the interface, $\therefore p = p'$)

$$\Rightarrow g\eta(\rho-\rho') = \frac{1}{2}\rho' q_0'^2 - \frac{1}{2}\rho q_0^2$$

$$= \frac{1}{2}\rho'(V'-c)^2\left[1+2m\eta\coth mh'\right]$$

$$- \frac{1}{2}\rho(V-c)^2\left[1-2m\eta\coth mh\right]$$

comparing the coefficient of η on both sides, we get,

$$g(\rho-\rho') = m\rho'(V'-c)^2\coth mh' + m\rho(V-c)^2\coth mh$$

This is the required condition for determining the velocity of propagation c of waves of length $\lambda = \frac{2\pi}{m}$ at the interface.

Case (I):

If the liquids are at rest i.e. $V=0$, $V'=0$, then from (11),

$$g(\rho-\rho') = m\rho' c^2 \coth mh' + m\rho c^2 \coth mh$$

$$\Rightarrow c^2 = \frac{g(\rho-\rho')}{m(\rho' \coth mh + \rho' \coth mh')}.$$

Case(II): Let the fluids be at rest so that $v=v'=0$ and the upper liquid be air of specific gravity $\frac{\rho'}{\rho}$ and if infinite depth so that

$\coth mh' \to 1$ as $h' \to \infty$ then,

$$c^2 = \frac{g}{m}\,\frac{\rho(1-\sigma)}{\rho(\coth mh' + \sigma)}$$

$$\Rightarrow c^2 = \frac{g}{m}\,\frac{1-\sigma}{\coth mh}\left(1+\sigma \tanh mh\right)^{-1}$$

$$= \frac{g(1-\sigma)}{m}\tanh mh\left(1-\sigma \tanh mh\right);$$ neglecting the higher order terms

$$= \frac{g}{m}\tanh mh\left\{1-\sigma \tanh mh - \sigma\right\}$$

$$= \frac{g}{m}\tanh mh\left\{1-\frac{\rho'}{\rho}(1+\tanh mh)\right\},$$ approx.

Case(III): Let the fluids be at rest so that $v=v'=0$ and the depth of both liquids be so large compared to the wave length that we may take

$\coth mh = \coth mh' = 1$

$$\therefore c^2 = \frac{g(\rho-\rho')}{m(\rho+\rho')} = \frac{g\lambda}{2\pi}\left[\frac{\rho-\rho'}{\rho+\rho'}\right]; \quad \because \lambda = \frac{2\pi}{m}$$

Case(IV): If the velocities v, v' make angles α, α' with the direction of c, then we have,

$$g(\rho-\rho') = m\rho'(v'\cos\alpha' - c)^2 \coth mh' + m\rho(v\cos\alpha - c)^2 \coth mh.$$

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HW

Ex. An infinite liquid of density σ lies above an infinite liquid of density ρ, the two liquids being separated by a horizontal plane of interface such that the velocity v of propagation of wave length λ along the interface is given by $v^2 = \frac{g\lambda}{2\pi}\left(\frac{\rho-\sigma}{\rho+\sigma}\right)$, provided $\rho > \sigma$.