$\Rightarrow \vec{r} \cdot \vec{h} = 0 \longrightarrow (12)$, $[\vec{r} \cdot (\vec{r} \times \vec{v}) = [\vec{r}, \vec{r}, \vec{v}] = 0$

The eqn. (12) represents a plane through S perpendicular to the vector $\vec{h}$.

$\vec{h} = h_1\hat{i} + h_2\hat{j} + h_3\hat{k}$

Thus the orbit of the planet (or satellite) arround the sun (or Earth) is a plane curve.

Solution of the Two-Body Problem

The Eqn of motion of the two body problem is $\ddot{\vec{r}} = -\frac{\mu \hat{r}}{r^2} \longrightarrow (1)$

Taking vector product of equation with $\vec{h}$, we get $\vec{h} \times \ddot{\vec{r}} = -\frac{\mu}{r^3}(\vec{h} \times \vec{r})$

$$= -\frac{\mu}{r^3}(\vec{r} \times \dot{\vec{r}}) \times \vec{r}$$

$[\because \vec{h} = \vec{r} \times \vec{v}]$

$$= -\frac{\mu}{r^3}\left[(\vec{r} \cdot \vec{r})\dot{\vec{r}} - (\vec{r} \cdot \dot{\vec{r}})\vec{r}\right]$$

$$= -\frac{\mu}{r^3}\left[r^2\dot{\vec{r}} - (r\dot{r})\vec{r} \right]$$

$$= -\mu\left(\frac{\dot{\vec{r}}}{r} - \frac{\vec{r}\cdot\dot{r}}{r^2} \right) \quad [\because \vec{r}\cdot\dot{\vec{r}} = r\dot{r}]$$

$$\Rightarrow -\frac{d}{dt}(\dot{\vec{r}}\times\vec{h}) = -\mu\cdot\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

$$\Rightarrow \frac{d}{dt}(\dot{\vec{r}}\times\vec{h}) = \mu\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

Integrating, $\dot{\vec{r}}\times\vec{h} = \mu\frac{\vec{r}}{r} + \vec{p}$,

where $\vec{p}$ is the vector constant of integration,

Now taking the dot product of this eqn with $\vec{r}$, we obtain the following eqn

$$\vec{r}\cdot(\dot{\vec{r}}\times\vec{h}) = \vec{r}\cdot\mu\frac{\vec{r}}{r} + \vec{r}\cdot\vec{p}$$

$$\Rightarrow (\vec{r}\times\dot{\vec{r}})\cdot\vec{h} = \mu\frac{r^2}{r} + \vec{r}\cdot\vec{p}$$

$$\Rightarrow \vec{h}\cdot\vec{h} = \mu r + rp\cos\theta \quad [\because \vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$
and $\vec{a}\cdot\vec{a} = a^2$

where α is the angle between the constant vector $\vec{p}$ and the radius vector $\vec{r}$.

$$\Rightarrow h^2 = \mu r + r p \cos\alpha$$

$$\Rightarrow r = \frac{h^2}{\mu + p\cos\alpha}$$

$$= \frac{h^2/\mu}{1 + \frac{p}{\mu}\cos\alpha}$$, which represent a conic with eccentricity..

$$e = \frac{p}{\mu}$$ and semi-latus rectum $$l = \frac{h^2}{\mu}$$

It is an ellipse if $e < 1$

a Hyperbola if $e > 1$ and

(a) parabola if $e = 1$

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