$$\mathbf{v}^{(2)} = \mathbf{D}^{-1}\,\mathbf{r}^{(2)} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} -0.5335 \\ -0.9502 \\ -0.7501 \end{bmatrix} = \begin{bmatrix} -0.1334 \\ -0.1900 \\ -0.2500 \end{bmatrix}$$

$$\mathbf{x}^{(3)} = \mathbf{x}^{(2)} + \mathbf{v}^{(2)} = [0.9333, -1.0733, -1.1000]^T.$$

Note that we obtain the same result from both the techniques.

Gauss-Seidel Iteration Method

We now use on the right hand side of (3.69), all the available values from the present iteration. We write the **Gauss-Seidel** method as

$$x_1^{(k+1)} = -\frac{1}{a_{11}}\left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \cdots + a_{1n}x_n^{(k)}\right) + \frac{b_1}{a_{11}}$$

$$x_2^{(k+1)} = -\frac{1}{a_{22}}\left(a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)} + \cdots + a_{2n}x_n^{(k)}\right) + \frac{b_2}{a_{22}}$$

$$\vdots$$

$$x_n^{(k+1)} = -\frac{1}{a_{nn}}\left(a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} + \cdots + a_{n,\,n-1}x_{n-1}^{(k+1)}\right) + \frac{b_n}{a_{nn}}$$

which may be rearranged in the form

$$a_{11}x_1^{(k+1)} = -\sum_{i=2}^{n} a_{1i}\,x_i^{(k)} + b_1$$

$$a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} = -\sum_{i=3}^{n} a_{2i}x_i^{(k)} + b_2$$

$$\vdots$$

$$a_{n1}x_1^{(k+1)} + \cdots + a_{nn}x_n^{(k+1)} = b_n. \tag{3.72}$$

Since we replace the vector $\mathbf{x}^{(k)}$ in the right side of (3.69) element by element, this method is also called the *method of successive displacement*.

In matrix notation, (3.72) becomes

$$(\mathbf{D} + \mathbf{L})\,\mathbf{x}^{(k+1)} = -\,\mathbf{U}\,\mathbf{x}^{(k)} + \mathbf{b}$$

or

$$\begin{aligned} \mathbf{x}^{(k+1)} &= -(\mathbf{D} + \mathbf{L})^{-1}\,\mathbf{U}\,\mathbf{x}^{(k)} + (\mathbf{D} + \mathbf{L})^{-1}\mathbf{b} \\ &= \mathbf{H}\,\mathbf{x}^{(k)} + \mathbf{c},\ k = 0, 1, 2, \ldots \end{aligned} \tag{3.73}$$

where

$$\mathbf{H} = -(\mathbf{D} + \mathbf{L})^{-1}\,\mathbf{U} \text{ and } \mathbf{c} = (\mathbf{D} + \mathbf{L})^{-1}\,\mathbf{b}.$$

Equation (3.73) can alternatively be written as

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} - [\mathbf{I} + (\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}]\,\mathbf{x}^{(k)} + (\mathbf{D} + \mathbf{L})^{-1}\mathbf{b} \\ &= \mathbf{x}^{(k)} - (\mathbf{D} + \mathbf{L})^{-1}\,(\mathbf{D} + \mathbf{L} + \mathbf{U})\,\mathbf{x}^{(k)} + (\mathbf{D} + \mathbf{L})^{-1}\mathbf{b} \\ &= \mathbf{x}^{(k)} - (\mathbf{D} + \mathbf{L})^{-1}\,\mathbf{A}\mathbf{x}^{(k)} + (\mathbf{D} + \mathbf{L})^{-1}\mathbf{b} \\ &= \mathbf{x}^{(k)} + (\mathbf{D} + \mathbf{L})^{-1}\,(\mathbf{b} - \mathbf{A}\mathbf{x}^{(k)}). \end{aligned}$$

We write

$$\mathbf{v}^{(k)} = (\mathbf{D} + \mathbf{L})^{-1}\,\mathbf{r}^{(k)}$$

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where $\mathbf{v}^{(k)} = \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}$ and $\mathbf{r}^{(k)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(k)}$ is the residual vector.

We may rewrite the above equations as

$$(\mathbf{D} + \mathbf{L})\,\mathbf{v}^{(k)} = \mathbf{r}^{(k)} \tag{3.74}$$

and solve for $\mathbf{v}^{(k)}$ by forward substitution. The solution is then found from

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{v}^{(k)}.$$

These equations describe the Gauss-Seidel method in an error format.

Example 3.22 Solve the system of equations

$$2x_1 - x_2 + 0x_3 = 7$$
$$-x_1 + 2x_2 - x_3 = 1$$
$$0x_1 - x_2 + 2x_3 = 1$$

using the Gauss-Seidel method given in equations (3.73) and its error format given in equations (3.74). Take the initial approximation as $\mathbf{x}^{(0)} = \mathbf{0}$ and perform three iterations.

(i) We have

$$\mathbf{D} + \mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The Gauss-Seidel method gives

$$\mathbf{x}^{(k+1)} = -(\mathbf{D} + \mathbf{L})^{-1}\,\mathbf{U}\,\mathbf{x}^{(k)} + (\mathbf{D} + \mathbf{L})^{-1}\mathbf{b}.$$

We get

$$(\mathbf{D} + \mathbf{L})^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix}$$

$$(\mathbf{D} + \mathbf{L})^{-1}\,\mathbf{U} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & 0 \\ 0 & -1/4 & -1/2 \\ 0 & -1/8 & -1/4 \end{bmatrix}$$

$$(\mathbf{D} + \mathbf{L})^{-1}\,\mathbf{b} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 9/4 \\ 13/8 \end{bmatrix}$$

Therefore, we obtain the iteration scheme

$$\mathbf{x}^{(k+1)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} \mathbf{x}^{(k)} + \begin{bmatrix} 7/2 \\ 9/4 \\ 13/8 \end{bmatrix}$$

Chapter 3

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Starting with zero initial vector, we get

$$\mathbf{x}^{(1)} = \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}, \mathbf{x}^{(2)} = \begin{bmatrix} 4.625 \\ 3.625 \\ 2.3125 \end{bmatrix}, \text{ and } \mathbf{x}^{(3)} = \begin{bmatrix} 5.3125 \\ 4.3125 \\ 2.6563 \end{bmatrix}$$

The exact solution is $\mathbf{x} = [6, 5, 3]^T$.

(ii) Using (3.74), we get for $\mathbf{x}^{(0)} = \mathbf{0}$

$k = 0$: $\mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\,\mathbf{x}^{(0)} = [7, 1, 1]^T$

$$\mathbf{v}^{(0)} = (\mathbf{D} + \mathbf{L})^{-1}\,\mathbf{r}^{(0)} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}$$

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \mathbf{v}^{(0)} = [3.5, 2.25, 1.625]^T$$

$k = 1$: $$\mathbf{r}^{(1)} = \mathbf{b} - \mathbf{A}\,\mathbf{x}^{(1)} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix} = \begin{bmatrix} 2.25 \\ 1.625 \\ 0 \end{bmatrix}$$

$$\mathbf{v}^{(1)} = (\mathbf{D} + \mathbf{L})^{-1}\,\mathbf{r}^{(1)} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 2.25 \\ 1.625 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.125 \\ 1.375 \\ 0.6875 \end{bmatrix}$$

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \mathbf{v}^{(1)} = [4.625, 3.625, 2.3125]^T$$

$k = 2$: $$\mathbf{r}^{(2)} = \mathbf{b} - \mathbf{A}\,\mathbf{x}^{(2)} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 4.625 \\ 3.625 \\ 2.3125 \end{bmatrix} = \begin{bmatrix} 1.375 \\ 0.6875 \\ 0 \end{bmatrix}$$

$$\mathbf{v}^{(2)} = (\mathbf{D} + \mathbf{L})^{-1}\,\mathbf{r}^{(2)} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 1.375 \\ 0.6875 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6875 \\ 0.6875 \\ 0.3438 \end{bmatrix}$$

$$\mathbf{x}^{(3)} = \mathbf{x}^{(2)} + \mathbf{v}^{(2)} = [5.3125, 4.3125, 2.6563]^T$$

Note that we obtain identical results by both the techniques.

Successive Over Relaxation (SOR) Method

This method is a generalization of the Gauss-Seidel method. This method is often used when the coefficient matrix of the system of equations is symmetric and has 'property *A*'. We define an auxiliary vector $\hat{\mathbf{x}}$ as

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