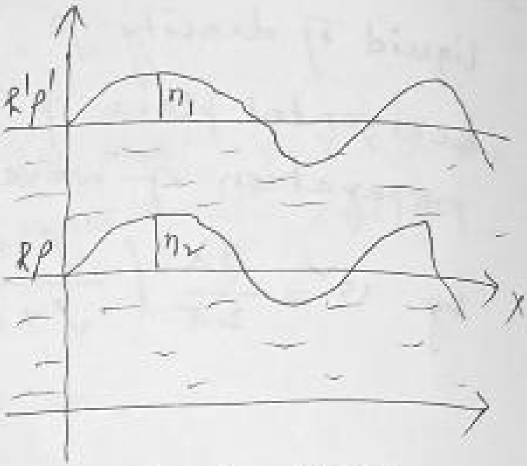
§: Waves at an interface with upper surface free:



Let a liquid of density ρ' and depth h' lie over another liquid of density ~~$h'\rho'$~~ ρ and depth h and let the liquids be at rest except for the wave motion. If the system be slightly disturbed then the waves at the common surface and the force surface can be propagated. Let us suppose a common velocity of wave propagation C at the free surface of the upper liquid and at the common surface. To make the motion, steady, we impose on the whole mass a velocity equal and opposite to that of propagation of waves, the wave profile gets fixed in space and the liquid begain to flow with velocity C in the $-ve$ direction of x-axis with choosen as the figure complex potentials of the lower and upper liquids are

$\eta_1 = a \sin mx$

$\Rightarrow \frac{\eta_1}{a} = \sin mx$

$$w = cz + \frac{ac}{\sinh mh} \cdot \cos m(z + ih) \longrightarrow (1)$$

$$\text{and } w' = cz + \frac{bc \cos mz}{\sinh mh'} - \frac{ac \cos m(z - ih')}{\sinh mh'} \longrightarrow (11)$$

The first expression is written from the consideration of steady motion value and for the second, we had to replace h by $-h'$ in (1) besides the addition of the term $\frac{bc}{\sinh mh'} \cos mz$ which represents the complex potential of a simple sine wave

$$\eta_2 = b \sin mx$$

at the instant $t=0$, Infact, (11) is obtained from the

Superposition of (1).

For the speed in the lower liquid, we have,

$$q^2 = \frac{dw}{dz} \cdot \frac{d\bar{w}}{dz}$$

$$= c^2\left[1 - \frac{am}{\sinh mh} \cdot \sin m(z+ih)\right] \times \left[1 - \frac{am \sin m(\bar{z} - ih)}{\sinh mh}\right]$$

$$\Rightarrow q^2 = c^2\left[1 - \frac{2am}{\sinh mh} \cdot \sin mx \cosh m(y+h)\right] \longrightarrow (3)$$

; [neglecting a^2]

For the speed in the upper liquid, we have,

$$q'^2 = \frac{dw'}{dz}\frac{d\bar{w}'}{dz} = c^2\left[1 - \frac{bm \sin mz}{\sinh mh'} + \frac{am \sin m(\bar{z} - ih')}{\sinh mh'}\right]$$

$$= c^2\left[1 - \frac{2b \sin mx \cosh my}{\sinh mh'} + \frac{2am \sin mx \cosh m(y-h')}{\sinh mh'}\right]$$

neglecting a^2, b^2, ab etc. To obtain the velocities q_0' and q_0 at the interface due to upper and lower liquids, we put,

$\eta_1 = a \sin mx$, $y(t) = 0$ in (3) & (4) and then

$$q_0^2 = c^2\left[1 - \frac{2am \sin mx \cosh mh}{\sinh mh}\right] \quad ; (\text{approx.})$$

$$= c^2\left[1 - 2m\eta_1 \coth mh\right]$$

and $$q_0'^2 = c^2\left[1 - \frac{2bm \sin mx}{\sinh mh'} + \frac{2am \sin mx \cosh mh'}{\sinh mh'}\right]$$

$$= c^2\left[1 + 2m\eta_1 \coth mh' - \frac{2mb\eta_1}{a \sinh mh'}\right], (\text{approx.})$$

At the common surface, the pressure equation for steady motion gives,

$$\frac{p'}{\rho'} + \frac{1}{2} q_0'^2 + g\eta_1 = \text{constant} \quad (\text{for upper Liquid})$$

$$\frac{p}{\rho} + \frac{1}{2} q_0^2 + g\eta_1 = \text{constant} \quad (\text{for lower Liquid})$$

But at the common surface $p = p'$, hence

$$(\rho' q_0'^2) - \rho q_0^2 + 2g\eta_1(\rho' - \rho) = \text{constant}.$$

Now, substituting the values of q_0^2 and $q_0'^2$ in the above expression, we get,

$$g\eta_1(\rho' - \rho) + \rho' c^2\left(\frac{1}{2} + m\eta_1 \coth mh' - \frac{mb\,\eta_1}{a \sinh mh'}\right)$$

$$-\rho c^2\left(\frac{1}{2} - m\eta_1 \coth mh\right) = \text{constant}.$$

This equation holds for every value of η_1 and so coefficient of η_1 must vanish

Thus we have,

$$\Rightarrow g(\rho - \rho') = c^2 m\left[\rho' \coth mh' + \rho \coth mh - \rho' \frac{1}{a} \operatorname{cosech} mh\right] \rightarrow (6)$$

Since p' is constant at the free surface

$$y = h' + b \sin mx = h' + \eta_2$$

Then pressure equation,

$$\frac{\cancel{p'}}{\rho'} + \frac{1}{2} q'^2 + gy = \text{constant}$$

becomes $\frac{1}{2} q'^2 \rho' + g\rho' y = \text{constant}.$

$$\text{or, } \frac{1}{2}\rho' c^2\left(1 - \frac{2bm \sin mx \cosh mh'}{\sinh mh'} + \frac{2am \sin mx}{\sinh mh'}\right)$$

$$+ g\rho'(h' + b \sin mx) = \text{constant}, \quad (\text{approx}).$$

As, before, the coefficient of $\sin mx$ is to be zero, which gives,

$$g = c^2 m(\coth mh' - \operatorname{cosech} mh')$$

$$\text{or, } \frac{b}{a} = \frac{c^2 m}{c^2 m \cosh mh' - g \sinh mh'} \longrightarrow (7)$$

Eliminating of $\frac{b}{a}$ between (6) & (7) yields

$$c^4 m^2(\rho \coth mh \cdot \coth mh' + \rho') - c^2 m \rho g(\coth mh + \coth mh') + g^2(\rho - \rho') = 0 \longrightarrow (8)$$

This equation gives two possible velocities of propagation for a given wave length whatever $\rho > \rho'$.

<u>Cor</u>: If the lower liquid is deep then $\coth mh = 1$ and eq^n (8) becomes,

$$c^4 m^2(\rho \coth mh' + \rho') - c^2 m g \rho(1 + \coth mh') + g^2(\rho - \rho') = 0$$

$$\Rightarrow c^4 m^2(\rho \coth mh' + \rho') - mc^2 g(\rho - \rho') - mc^2 g(\rho \coth mh' + \rho') + g^2(\rho - \rho') = 0$$

$$\Rightarrow mc^2\left[mc^2(\rho \coth mh' + \rho') - g(\rho \coth mh' + \rho')\right] - g(\rho - \rho')\left[mc^2 - g\right] = 0$$

$$\Rightarrow mc^2\left[(mc^2 - g)(\rho \coth mh' + \rho')\right] - g(\rho - \rho')(mc^2 - g) = 0$$

$$\Rightarrow (mc^2 - g)\left[mc^2(\rho \coth mh' + \rho') - g(\rho - \rho')\right] = 0$$

where $c^2 = \frac{g}{m}$ and $c^2 = \frac{g}{m} \frac{\rho - \rho'}{(\rho \coth mh' + \rho')}$.

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