

Which are the diff<sup>n</sup> eqns. of the elements of equal pressure & density?  $\left( \frac{dp}{\rho g} = -dx, \frac{dp}{\rho g} = -dy, \frac{dp}{\rho g} = -dz \right)$

- Q. A liquid of given volume V is to be stored under W.L.F. forces  $x = -\frac{ax}{a^2}, y = -\frac{ay}{b^2}$  and  $z = -\frac{az}{c^2}$  where  $a, b, c$  are constants. Find the pressure at any pt. of W.L.F. liquid and surface of equal pressure.

Sol<sup>n</sup> We have,  $(dx + dy + dz) = \frac{1}{\rho g} dp = dP$

$$x = -\frac{ax}{a^2}, y = -\frac{ay}{b^2}, z = -\frac{az}{c^2}$$

Pressure at any pt.  $(x, y, z)$  in given by,

$$\text{④ } \frac{dp}{\rho g} (dx + dy + dz) = dP$$

$$\Rightarrow -\mu g \left( \frac{x}{a^2} dx + \frac{y}{b^2} dy + \frac{z}{c^2} dz \right) = dP$$

Integrating, we get the pressure at any pt. of W.L.F.

$$\Rightarrow P_0 - \frac{\mu g}{2} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = P$$

The surface of zero pressure is the free surface.

$$\text{in given key } P = 0 \Rightarrow (z - 0) c = 0 \Rightarrow (z - 0)c = 0$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{2c}{\mu g}$$

This is an ellipsoid whose volume is given to be  $V$

$$\sqrt{2} \cdot \frac{4}{3} \pi abc \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)^{3/2}$$

$$= \frac{4}{3} \pi abc \left( \frac{2c}{\mu g} \right)^{3/2}$$

$$\Rightarrow c = \frac{\mu g}{2} \left( \frac{3v}{4\pi abc} \right)^{2/3}$$

increasing density decreases the effect of gravity

$$\therefore p = \frac{\mu g}{2} \left( \frac{3v}{4\pi abc} \right)^{2/3} - \frac{\mu g}{2} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)$$

This gives pressure at any pt. surfaces of equal pressure are obtained by putting

$p = \text{constant} = A$  (say) for finding surfaces of equal pressure

$\therefore$  Surfaces of equal pressure are -

$$A = c - \frac{1}{2} \rho \mu \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)$$

and similarly

$$\text{or}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{c}{A} = k^2 \quad \text{say}$$

Q. Prove that if the force per unit of mass of

$x, y, z$  parallel to the axes are  $y(a-z), x(a-z), xy$ . The surfaces of equal pressure are hyperboloids of one sheet and the curves of equal pressure and density are rectangular hyperbolae.

Soln As we have given us to express it

$$x = y(a-z); \quad y = x(a-z); \quad z = xy$$

We know,

$$x \left( \frac{\partial y}{\partial z} - \frac{\partial z}{\partial y} \right) + y \left( \frac{\partial z}{\partial x} - \frac{\partial x}{\partial z} \right) + z \left( \frac{\partial x}{\partial y} - \frac{\partial y}{\partial x} \right)$$

$$x \left( \frac{a}{z} - \frac{1}{a} + \frac{1}{a} - \frac{a}{z} \right) + y \left( \frac{1}{z} - \frac{1}{a} + \frac{1}{a} - \frac{1}{z} \right) + z \left( \frac{a}{y} - \frac{a}{y} \right) = 0$$

$$\therefore \left( \frac{\partial x}{\partial y} - \frac{\partial y}{\partial x} \right) = 0$$

So that,

$$\frac{\partial x}{\partial y} = a-z, \quad \frac{\partial x}{\partial z} = -y, \quad \frac{\partial y}{\partial z} = a-z$$

and hence the necessary loops for excess left graph

$$\frac{\partial y}{\partial z} = -x, \quad \frac{\partial z}{\partial x} = y, \quad \frac{\partial z}{\partial y} = x.$$

The condition of equilibrium is

$$x \left( \frac{\partial y}{\partial z} - \frac{\partial z}{\partial y} \right) + y \left( \frac{\partial z}{\partial x} - \frac{\partial x}{\partial z} \right) + z \left( \frac{\partial x}{\partial y} - \frac{\partial y}{\partial x} \right) = 0$$

$$\Rightarrow x(-n-n) + y(y+y) + z(a-z-a+z) = 0$$

$$\Rightarrow -2nx + 2y^2 + 0 = 0$$

$$\Rightarrow -2n[y(a-z)] + 2y[n(a-z)] = 0$$

$$\Rightarrow -2n[a + 2xyz + 2yna - 2yaz] = 0$$

$$\Rightarrow 0 = 0$$

which is clearly satisfied

Now, surfaces of equal pressure are given by

$P = \text{constant}$  which implies  $dP = 0$

$$\Rightarrow pdx + pdy + pdz = 0$$
$$\Rightarrow y(a-z)dx + n(a-z)dy + nydz = 0$$

( $\therefore$  putting value of  $x, y, z$ )

Dividing by  $ny(a-z)$ . The surfaces of equal pressure

are given by

$$\frac{dx}{n} + \frac{dy}{y} + \frac{dz}{a-z} = 0$$

Integrating,

$$\log n + \log y + \log(a-z) = \log C$$

$$\Rightarrow \frac{ny}{a-z} = c \Rightarrow ny = c(a-z)$$

which are clearly hyperbolic paraboloids.

Again, the curves of equal pressure & equal density are given by

$$\begin{aligned} \frac{dn}{\frac{\partial Y}{\partial Z} - \frac{\partial Z}{\partial Y}} &= \frac{dy}{\frac{\partial Z}{\partial X} - \frac{\partial X}{\partial Z}} = \frac{dz}{\frac{\partial X}{\partial Y} - \frac{\partial Y}{\partial X}} \\ \Rightarrow \frac{dn}{n-y} &= \frac{dy}{2y} = \frac{dz}{0} \\ \Rightarrow \frac{dn}{-2n} &= \frac{dy}{2y} = \frac{dz}{0} + (x+b)y + (a-w)z \\ \Rightarrow \frac{dn}{-n} &= \frac{dy}{y} - \frac{dz}{0} \quad 0 = a + b w + x + c - z \\ \text{The last fraction given } dz = 0 \\ \Rightarrow z &= \text{const.} \end{aligned}$$

and the 1st 2 fractions give  $\frac{dn}{n} + \frac{dy}{y} = 0$   
~~the 3rd will be zero, hence  $\Rightarrow$  my = const~~  
~~thus the curves of equal pressure & equal density~~  
~~are given by  $ny = \text{const}$~~   
~~hence  $ny = \text{const}$ ,  $z = \text{const}$ .~~

which are clearly rectangular hyperbola

$$y = \frac{zb}{5-b} + \frac{ab}{5-b} + \frac{xb}{4-b}$$

$$2ab = (z-a)b \quad \text{perpendicular}$$

Q. If  $x = y(y+z)$ ,  $y = z(z+u)$ ,  $z = u(u-y)$  integrate  
surfaces of equal pressure are the hyperbolic paraboloids.  
 $y(u+z) = c(y+z)$  & the curves of equal pressure & density  
given by  $y(u+z) = \text{const}$ ,  $y+z = \text{const}$ .

Soln We know that, ~~the surfaces of varying  $\rho_2$  will be like~~

$$dP = g [x du + y dy + z dz] \text{ Law of density}$$

Now, for the surfaces of equal pressure  $P = \text{const}$ .

$$\text{i.e., } dP = 0 \quad \frac{x}{u} - \frac{x}{v} = \frac{u}{v} - \frac{u}{w} = \frac{w}{v} - \frac{w}{u}$$

$$\therefore x du + y dy + z dz = 0$$

$$\Rightarrow y(y+z) du + z(z+u) dy + y(y-u) dz = 0$$

$$\Rightarrow \frac{du}{z+u} + \frac{z dy}{y(y+z)} + \frac{(y-u)}{(z+u)(y+z)} dz = 0$$

$$\Rightarrow \frac{dz}{z+u} + \frac{z dy}{y(y+z)} + \frac{(y+z)-(z+u)}{(z+u)(y+z)} dz = 0$$

$$\Rightarrow \frac{du}{z+u} + \frac{z dy}{y(y+z)} + \frac{dz}{z+u} = \frac{dz}{y+z} = 0$$

$$\Rightarrow \frac{du+dz}{z+u} + \frac{z dy - y dz}{y(y+z)} = 0$$

$$\Rightarrow \frac{dz+du}{z+u} + \frac{dy(z+y) - y(dz+dy)}{y(y+z)} = 0$$

$$\Rightarrow \frac{d(u+z)}{u+z} + \frac{dy}{y} + \frac{d(y+z)}{y+z} = 0$$

$$\therefore u+z = \rho_2(s+b) + (s+b+u+b) e^{-y}$$

Integrating,

$$(x+b)t = \frac{1}{2}, (x+s)t = \frac{1}{2}, (x+b)t = x + b$$

~~considering straight line curves with  $t = 1$~~   $\log(x+z) + \log y - \log(y+z) = \log \text{const}$  to satisfy

$$\Rightarrow \frac{(x+z)+y}{y+z} = C$$

which is the eq" giving surfaces of equal pressure.

Again, curves of equal pressure & density is given by

$$\frac{dx}{dt} = 0 \text{ gives straight lines to curves with const. work}$$
$$\frac{\frac{dy}{dz}}{\frac{\partial z}{\partial x} - \frac{\partial x}{\partial y}} = \frac{\frac{dz}{dx}}{\frac{\partial z}{\partial y} - \frac{\partial y}{\partial x}} = \frac{\frac{dy}{dx}}{\frac{\partial x}{\partial y} - \frac{\partial y}{\partial x}} \quad 0 = qb$$
$$0 = sbz + tbP + xbz \dots$$

$$\Rightarrow \frac{dx}{x+z-y} = \left( -\frac{dy}{y} + \frac{dz}{z} \right) \text{ const. } (s+t)b \dots \textcircled{1}$$
$$0 = sb \cdot \frac{(x-b)}{(x+s)(x+b)} + \frac{tbz}{(s+t)b} + \frac{xb}{x+s} \leftarrow$$

from the last two fractions

$$0 = sb \frac{(x+s)(x+b)}{(x+s)(x+b)} + \frac{tbz}{(s+t)b} + \frac{xb}{x+s} \leftarrow$$
$$dy + dz = 0 \Rightarrow y+z = \text{const.}$$

Also, using the multipliers,  $y \frac{bzb}{(x+z-y)}, y \frac{bzb}{(x+y)}$  respectively we have,

$$\frac{ydx}{y(x+z-y)} + \frac{(x+z)dy}{y(x+z-y)} + \frac{ydz}{y^2} = 0$$
$$\Rightarrow ydx + (x+z)dy + \frac{ydz}{x+s} = 0$$
$$\Rightarrow ydx + xdy + zdy + ydz = 0 + \frac{(s+t)b}{x+s} \leftarrow$$
$$\Rightarrow y(dx+dy) + zdy + ydz = 0 + \frac{(s+t)b}{x+s} \leftarrow$$
$$\Rightarrow y(dx+dy) + (x+z)dy = 0$$

$$\Rightarrow y(n+z)(x^2 + pb^2 + xb^2) = \frac{1}{n+z} \quad \text{Eq. 1}$$

$\Rightarrow \frac{d(n+z)}{n+z} + \frac{dy}{y} = 0$

Integrating both sides  $\Rightarrow \ln(n+z) + pb^2 + xb^2 = C$

$$\Rightarrow \log(n+z) + \log y = \log e$$

$$\Rightarrow (n+z)y = \text{const} \rightarrow \text{Eq. 2}$$

$$g \rho_0 + r \frac{1}{A} = g \rho_0 \text{, for } \rho_0$$

① and ② together represent curves of equal pressure & density.

Boyle's law &  
Charles law

\* Elastic fluid: tends to反抗 & returning back at

Q A mass of elastic fluid is at rest under the action of given forces. Determine the pressure at any pt for both the cases when the thm. remains constant and when it varies.

Soln When thm (in constant):  $\frac{q}{TA} = q_b$

We have  $(s_b s + pb^2 + xb^2) \frac{1}{TA} = \frac{q_b}{q}$

$$p = k p \quad (\text{Boyle's law})$$

$$\text{and } dp = p(xdu + ydy + zdz) \text{ with all } \\ = \frac{p}{k} (xdu + ydy + zdz)$$