which are the diff^n eq^{n}s. of the curves of equal pressure & density.

2010, 14, 4

Q: A liquid of given volume V is at rest under the forces $X = -\frac{\mu x}{a^2}$, $Y = -\frac{\mu y}{b^2}$, $Z = -\frac{\mu z}{c^2}$. Find the pressure at any pt. of the liquid and surfaces of equal pressure.

Sol^n We have,

$$X = -\frac{\mu x}{a^2}, \quad Y = \frac{-\mu y}{b^2}, \quad Z = -\frac{\mu z}{c^2}$$

∴ pressure at any pt. (x, y, z) is given by.

$$\rho(X\,dx + Y\,dy + Z\,dz) = dp \quad (*)$$

$$\Rightarrow -\mu\rho\left(\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy + \frac{z}{c^2}\,dz\right) = dp$$

Integrating,

$$\Rightarrow c - \frac{\mu\rho}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = p$$

The surface of zero pressure i.e, the free surface is given by $p = 0$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{2c}{\mu\rho}$$

This is an ellipsoid whose vol^m is given to be V

$$V = \frac{4}{3}\pi abc\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^{3/2}$$

$$= \frac{4}{3}\pi abc\left(\frac{2c}{\mu\rho}\right)^{3/2}.$$

$$\Rightarrow C = \frac{\mu \rho}{2} \left(\frac{3V}{4\pi abc}\right)^{2/3}$$

$$\therefore p = \frac{\mu \rho}{2} \left(\frac{3V}{4\pi abc}\right)^{2/3} - \frac{\mu \rho}{2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

This gives pressure at any pt. surfaces of equal pressure are obtain by putting

$p =$ constant $= A$ (say)

$\therefore$ Surfaces of equal pressure are -

$$A = C - \frac{1}{2} \rho \mu \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

or, $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 2(C-A)\frac{1}{\rho \mu} = k^2 \text{ (say)}$$

2015 Q: Prove that if the force per unit of mass at x, y, z parallel to the axes are $y(a-z)$, $x(a-z)$, xy. The surfaces of equal pressure are hyperbolic paraboloids and the curves of equal pressure and density are rectangular hyperbolas.

Soln As we have given

$$X = y(a-z); \quad Y = x(a-z); \quad Z = xy$$

We know,

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0$$

So that,

$$\frac{\partial X}{\partial y} = a-z, \quad \frac{\partial X}{\partial z} = -y, \quad \frac{\partial Y}{\partial x} = a-z$$

$$\frac{\partial Y}{\partial z} = -x, \quad \frac{\partial Z}{\partial x} = y, \quad \frac{\partial Z}{\partial y} = x.$$

The condition of equilibrium is

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0$$

$$\Rightarrow X(-x-x) + Y(y+y) + Z(a-z-a+z) = 0$$

$$\Rightarrow -2xX + 2yY + 0 = 0.$$

$$\Rightarrow -2x\,[y(a-z)] + 2y\,[x(a-z)] = 0$$

$$\Rightarrow -2x\cancel{y}a + 2\cancel{x}yz + 2\cancel{y}xa - 2y\cancel{x}z = 0$$

$$\Rightarrow 0 = 0$$

which is clearly satisfied

Now, surfaces of equal pressure are given by $p =$ constant which implies $dp = 0$

$$\Rightarrow X\,dx + Y\,dy + Z\,dz = 0$$

$$\Rightarrow y(a-z)\,dx + x(a-z)\,dy + xy\,dz = 0$$

($\because$ putting value of X, Y, Z)

Dividing by $xy(a-z)$. The surfaces of equal pressure are given by

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{a-z} = 0$$

Integrating.

$$\log x + \log y - \log(a-z) = \log c$$

$$\Rightarrow \frac{xy}{a-z} = c \quad \Rightarrow xy = c(a-z)$$

which are clearly hyperbolic paraboloids.

Again, the curves of equal pressure & equal density are given by

$$\frac{dx}{\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}} = \frac{dy}{\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}} = \frac{dz}{\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}}$$

$$\Rightarrow \frac{dx}{-x-x} = \frac{dy}{2y} = \frac{dz}{0}$$

$$\Rightarrow \frac{dx}{-2x} = \frac{dy}{2y} = \frac{dz}{0}$$

$$\Rightarrow \frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{0}$$

The last fraction gives $dz = 0$

$\Rightarrow z = \text{const.}$

and ~~~~ the 1st 2 fractions gave $\frac{dx}{x} + \frac{dy}{y} = 0$

$\Rightarrow xy = \text{const}$.

Thus the curves of equal pressure & equal density are given by

$xy = \text{const}$, $z = \text{const}$.

which are clearly rectangular hyperbola

Q. If $X = y(y+z)$; $Y = z(z+x)$, $Z = y(y-x)$ surfaces of equal pressure are the hyperbolic parabolas. $y(x+z) = c(y+z)$ & the curves of equal pressure & density given by $y(x+z) = \text{const}$, $y+z = \text{const}$.

Soln We know that, ~~X~~

$$dp = \rho\left[X\,dx + Y\,dy + Z\,dz\right]$$

Now, for the surfaces of equal pressure $p = \text{const}$.

i.e, $dp = 0$

$$\therefore X\,dx + Y\,dy + Z\,dz = 0$$

$$\Rightarrow y(y+z)\,dx + z(z+x)\,dy + y(y-x)\,dz = 0$$

$$\Rightarrow \frac{dx}{z+x} + \frac{z\,dy}{y(y+z)} + \frac{(y-x)}{(z+x)(y+z)}\,dz = 0$$

$$\Rightarrow \frac{dx}{z+x} + \frac{z\,dy}{y(y+z)} + \frac{(y+z)-(z+x)}{(z+x)(y+z)}\,dz = 0$$

$$\Rightarrow \frac{dx}{z+x} + \frac{z\,dy}{y(y+z)} + \frac{dz}{z+x} - \frac{dz}{y+z} = 0$$

$$\Rightarrow \frac{dx+dz}{z+x} + \frac{z\,dy - y\,dz}{y(y+z)} = 0$$

$$\Rightarrow \frac{dz+dx}{z+x} + \frac{dy\,(z+y) - y(dz+dy)}{y(y+z)} = 0$$

$$\Rightarrow \frac{d(x+z)}{x+z} + \frac{dy}{y} - \frac{d(y+z)}{y+z} = 0$$

Integrating,

$$\log(x+z) + \log y - \log(y+z) = \log c$$

$$\Rightarrow \frac{(x+z)+y}{y+z} = c$$

which is the eqn giving surfaces of equal pressure.

Again, curves of equal pressure & density is given by

$$\frac{dx}{\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}} = \frac{dy}{\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}} = \frac{dz}{\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}}$$

$$\Rightarrow \frac{dx}{x+z-y} = \frac{dy}{-y} = \frac{dz}{y} \rightarrow (1)$$

from the last two fractions

$$dy + dz = 0 \Rightarrow y+z = \text{const}.$$

Also, using the multipliers y, $x+z$, y respectively we have

$$\frac{y\,dx + (x+z)\,dy + y\,dz}{y(x+z-y) + \{-y(x+z)\} + y^2} = 0$$

$$\Rightarrow y\,dx + (x+z)\,dy + y\,dz = 0$$

$$\Rightarrow y\,dx + x\,dy + z\,dy + y\,dz = 0$$

$$\Rightarrow y(dx+dz) + (x+z)\,dy = 0$$

$$\Rightarrow y(x+z)(z+x)$$

$$\Rightarrow \frac{d(x+z)}{x+z} + \frac{dy}{y} = 0$$

$$\Rightarrow \log(x+z) + \log y = \log c$$

$$\Rightarrow (x+z)\,y = \text{const} \rightarrow (2)$$

(1) and (2) together represent curves of equal pressure & density.

3 24

Boil's law & charls law

* Elastic fluid:

2018 Q A mass of elastic fluid is at rest under the action of given forces. Determine the pressure at any pt for both the cases when the thm remains constant and when it varies.

Soln When thm is constant:

We have,

$$p = k\rho \quad (\text{Boyle's law})$$

and

$$dp = \rho(X\,dx + Y\,dy + Z\,dz)$$

$$= \frac{p}{k}(X\,dx + Y\,dy + Z\,dz)$$