

Question 18:

If $A = \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that

$$I + A = (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

Answer

On the L.H.S.

$$\begin{aligned} I + A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix} \quad \dots(1) \end{aligned}$$

On the R.H.S.

$$\begin{aligned} (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix} \right) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha + \sin \alpha \tan \frac{\alpha}{2} & -\sin \alpha + \cos \alpha \tan \frac{\alpha}{2} \\ -\cos \alpha \tan \frac{\alpha}{2} + \sin \alpha & \sin \alpha \tan \frac{\alpha}{2} + \cos \alpha \end{bmatrix} \quad \dots(2) \end{aligned}$$

On the L.H.S.

$$\begin{aligned} I + A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix} \quad \dots(1) \end{aligned}$$

On the R.H.S.

$$\begin{aligned} (I - A) &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix} \right) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha + \sin \alpha \tan \frac{\alpha}{2} & -\sin \alpha + \cos \alpha \tan \frac{\alpha}{2} \\ -\cos \alpha \tan \frac{\alpha}{2} + \sin \alpha & \sin \alpha \tan \frac{\alpha}{2} + \cos \alpha \end{bmatrix} \quad \dots(2) \end{aligned}$$

$$\begin{aligned} &= \begin{bmatrix} 1 - 2\sin^2 \frac{\alpha}{2} + 2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \tan \frac{\alpha}{2} & -2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} + \left(2\cos^2 \frac{\alpha}{2} - 1\right) \tan \frac{\alpha}{2} \\ -\left(2\cos^2 \frac{\alpha}{2} - 1\right) \tan \frac{\alpha}{2} + 2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} & 2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \tan \frac{\alpha}{2} + 1 - 2\sin^2 \frac{\alpha}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 - 2\sin^2 \frac{\alpha}{2} + 2\sin^2 \frac{\alpha}{2} & -2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} + 2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} - \tan \frac{\alpha}{2} \\ -2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} + \tan \frac{\alpha}{2} + 2\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} & 2\sin^2 \frac{\alpha}{2} + 1 - 2\sin^2 \frac{\alpha}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix} \end{aligned}$$

Thus, from (1) and (2), we get L.H.S. = R.H.S.