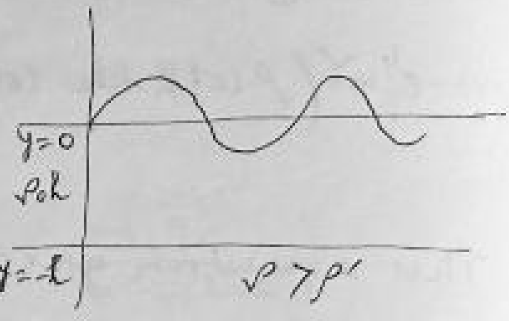
Ex. If there be two liquids in a straight canal of uniform sections of densities σ_1, σ_2 and depths l_1, l_2, show that the velocity of propagation of long waves is given by the equation

$$\left(\frac{c^2}{l_1 g}-1\right)\left(\frac{c^2}{l_2 g}-1\right)=\frac{\sigma_1}{\sigma_2}$$

where $\sigma_2 > \sigma_1$ and it is assumed that the liquids do not mixed.

Sol



Let a liquid of density ρ' and depth h' be over another liquid of density ρ and depth h and let the liquids be at rest.

If the system be slightly ~~distributed~~ distrubed, then the waves at the common surface and the free surface may be propagated. Let us consider a common velocity of waves propagation c at the free surface of the upper liquid and at the common surface. To make the motion steady, we impose on the whole mass a velocity equal and opposite to that of propagation of waves, the wave profile gets fixed in space and liquid begin to flow with velocity c in the negative direction of x-axis. We choose the axes as shown in figure.

Since the two liquids are contained in a straight canal and so the surface of the upper liquid must be free surface. In this situation, the equation which gives two possible velocities of propagation for a given wave length $\rho > \rho'$ is given by

$$c^4 m^2 (\rho \coth mh \coth mh' + \rho') - c^2 m g \rho (\coth mh + \coth mh') + g^2 (\rho - \rho') = 0 \longrightarrow (1), \text{ where } \rho > \rho'$$

In this problem,

$$\rho = \sigma_2, \; \rho' = \sigma_1, \; h = l_2, \; h' = l_1$$

Thus (1) becomes,

$$c^4 m^2 (\sigma_2 \coth ml_2 \cdot \coth ml_1 + \sigma_1) - c^2 m g \sigma_2 (\coth ml_2 + \coth ml_1) + g^2 (\sigma_2 - \sigma_1) = 0 \longrightarrow (11)$$

Since the curves formed at the common surface are long waves, hence

$$ml_1 \to 0, \; ml_2 \to 0 \quad \left(\text{for } \frac{h}{\lambda} = \frac{mh}{2\pi} \to 0\right)$$

consequently, $\tanh ml_2 = ml_2$, $\tanh ml_1 = ml_1$

Thus we have, $\coth ml_2 = \frac{1}{ml_2}$, $\coth ml_1 = \frac{1}{ml_1}$

Hence, (11) becomes,

$$c^4 m^2 \left(\frac{\sigma_2}{ml_2} \cdot \frac{1}{ml_1} + \sigma_1 \right) - c^2 m g \sigma_2 \left(\frac{1}{ml_2} + \frac{1}{ml_1} \right) + g^2 (\sigma_2 - \sigma_1) = 0$$

$$\text{or, } \frac{c^4 \sigma_2}{l_1 l_2} + c^4 m^2 \sigma_1 - c^2 \left(\frac{1}{l_2} + \frac{1}{l_1} \right) g \sigma_2 + g^2 (\sigma_2 - \sigma_1) = 0$$

Dividing both sides by σ_2, we get,

$$\frac{c^4}{l_1 l_2} + \frac{\sigma_1}{\sigma_2} c^4 m^2 - c^2 \left(\frac{1}{l_2} + \frac{1}{l_1} \right) g + g^2 \left(1 - \frac{\sigma_1}{\sigma_2} \right) = 0$$

In case of long waves, λ is very large and $m = \frac{2\pi}{\lambda}$. So m^2 may be neglected. Thus we have,

$$\frac{c^4}{l_1 l_2} - c^2 g \left(\frac{1}{l_1} + \frac{1}{l_2} \right) + g^2 \left(1 - \frac{\sigma_1}{\sigma_2} \right) = 0$$

$$\Rightarrow \frac{c^4}{l_1 l_2 g^2} - \frac{c^2}{g} \left(\frac{1}{l_1} + \frac{1}{l_2} \right) + 1 = \frac{\sigma_1}{\sigma_2}$$

$$\Rightarrow \frac{c^2}{\lambda_1 g}\left(\frac{c^2}{\lambda_2 g}-1\right)-1\left(\frac{c^2}{\lambda_2 g}-1\right)=\frac{\sigma_1}{\sigma_2}$$

$$\Rightarrow \left(\frac{c^2}{\lambda_1 g}-1\right)\left(\frac{c^2}{\lambda_2 g}-1\right)=\frac{\sigma_1}{\sigma_2} \quad \text{where } \sigma_2 > \sigma_1$$

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§. Group Velocity:

When waves are started by a local disturbance such as dropping of a stone into a canal or the motion of a boat through water, the successive wave have different length and are propagated with different velocities. Since the velocity of propagation of a simple harmonic train varies with the wave length so the waves of slightly different wave lengths will be sorted out into group. The velocity with which an isolated group of waves of considerably the same length, advances over relatively deep waters is called the group velocity.

We consider the case of two systems of simple harmonic waves of the same amplitude and of nearly but not quite the same wave lengths such that

$$\eta_1 = a\sin(mx-nt) \longrightarrow (i)$$

$$\eta_2 = a\sin\left[(m+\delta m)x-(n+\delta n)t\right] \longrightarrow (ii)$$

where δm and δn are small quantities.

§: Group Velocity:

We consider the case of two systems of simple harmonic waves of the same amplitude and of nearly but not quite the same wave length such that

$$\eta_1 = a\sin(mx - nt) \longrightarrow (\mathrm{I})$$

$$\eta_2 = a\sin\{(m+\delta m)x - (n+\delta n)t\} \longrightarrow (\mathrm{II})$$

where δm and δn are infinitesimal small quantities.

The resultant disturbance is given by

$$\eta = \eta_1 + \eta_2 = a\sin(mx - nt) + a\sin[(m+\delta m)x - (n+\delta n)t]$$

$$= 2a\cos\frac{x\delta m - t\delta n}{2}\cdot\sin\left\{\left(m+\frac{\delta m}{2}\right)x - \left(n+\frac{\delta n}{2}\right)t\right\}$$

$$= A\sin(m'x - n't) \longrightarrow (\mathrm{III})$$

where $A = 2a\cos\dfrac{x\delta m - t\delta n}{2}$, $m' = m + \dfrac{\delta m}{2}$, $n' = n + \dfrac{\delta n}{2}$ $\rightarrow$ (IV)

Hence the resultant wave is of the same form as one of the original waves but with a different amplitude. From (II), it follows that the amplitude of the resulting disturbance (II) varies as a wave velocity.

$$c_g = \frac{\delta n}{\delta m}, \text{ known as group velocity.}$$

In different notation, $c_g = \dfrac{dn}{dm}$

But wave velocity $c = \dfrac{n}{m}$

$\therefore n = cm$

Hence, $c_g = \dfrac{d}{dm}(cm)$

$$= c + m\frac{dc}{dm} \longrightarrow (\mathrm{V})$$

For the waves on the surface of water of depth h, we have the relation

$$c^2 = \frac{n^2}{m^2} = \frac{g}{m}\tanh mh.$$

Taking log on both sides, we get,

$$2\log c = \log g - \log m + \log \tanh mh$$

Differentiating w.r.t. to 'm', we have

$$\frac{2}{c}\frac{dc}{dm} = -\frac{1}{m} + h\frac{\operatorname{sech}^2 mh}{\tanh mh}$$

$$= -\frac{1}{m} + h \cdot \frac{1}{\cosh^2 mh} \cdot \frac{\cosh mh}{\sinh mh}$$

$$\Rightarrow m\frac{dc}{dm} = -\frac{c}{2} + \frac{cmh}{2} \cdot \frac{1}{\sinh mh \cdot \cosh mh}$$

$$\Rightarrow m\frac{dc}{dm} = -\frac{c}{2} + \frac{cmh}{\sinh 2mh}$$

Thus (v) becomes,

$$c_g = c + m\frac{dc}{dm}$$

$$= c - \frac{c}{2} + \frac{cmh}{\sinh 2mh}$$

$$= \frac{c}{2} + \frac{cmh}{\sinh 2mh}$$

$$= \frac{c}{2}\left[1 + \frac{2mh}{\sinh 2mh}\right] \longrightarrow (vi)$$

i.e. $$\frac{c_g}{c} = \frac{1}{2}\left[1 + \frac{2mh}{\sinh 2mh}\right] \longrightarrow (vii)$$

Case (i): for deep water, $h \to \infty$ so that $\frac{2mh}{\sinh 2mh} \to 0$.

Hence (vi) becomes

$$\frac{c_g}{c} = \frac{1}{2} \Rightarrow c_g = \frac{1}{2}c.$$

i.e. the group velocity for deep water is half the wave velocity.

Case (ii) : For shallow water, $h \to 0$ so that $\frac{2mh}{\sinh 2mh} \to 1$

$$\therefore (vi) \Rightarrow \frac{c_g}{c} = \frac{1}{2}(1+1) \Rightarrow c_g = c.$$

i.e. the group velocity for shallow water is equal to the wave velocity.

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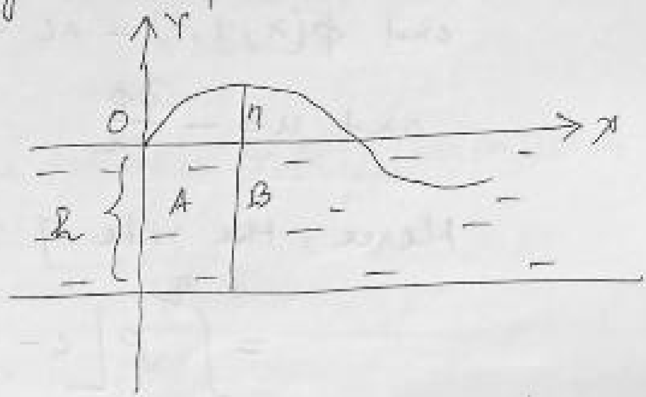
§: Dynamical significance of group velocity
(Rate of transmission of energy in simple harmonic surface waves).



When wave progresses it also carries energy along with it. For the wave in one-dimensional transmission of energy across a vertical wall is the rate of work done by pressure. Hence the rate of transmission of energy is measured by taking a vertical section of the liquid at right angles to the direction of propagation and determining the rate at which the liquid on one side of this section is doing work on the liquid of the other side. Let the two portions of the liquid be A and B, h the depth of liquid.

Now, the rate of transmission of energy

= Rate of work done by the region A on the region B

$$= \frac{\text{Force} \times \text{distance}}{\text{time}}$$

$$= \int_{-h}^{\eta} (p\,dy)\,u \text{ , as thrust on element } dy \text{ is}$$

$p\,dy$ and u is horizontal component of velocity.

In particular, let us consider the flow of energy accross a vertical wall in a rectangular channel of depth h when there is a simple harmonic progressive wave.

For this,

$$\eta(x,t) = a\sin(mx-nt) \longrightarrow (I)$$

and
$$\phi(x,y,t) = ac\,\frac{\cosh m(y+h)}{\sinh mh}\cos(mx-nt) \longrightarrow (II)$$

and
$$u = -\frac{\partial\phi}{\partial x}$$

Hence, the rate of transmission of energy

$$= \int_{-h}^{\eta} \rho\left[c - \frac{1}{2}q^2 + \frac{\partial p}{\partial t} - gy\right]\left(-\frac{\partial\phi}{\partial x}\right)dy \longrightarrow (III)$$

where the pressure equation

$$\frac{p}{\rho} + \frac{1}{2}q^2 - \frac{\partial\phi}{\partial t} + gy = c$$

$$= \int_{-h}^{\eta} \rho\left(c + \frac{\partial\phi}{\partial t} - gy\right)\left(-\frac{\partial\phi}{\partial x}\right)dy, \text{ neglecting the small quantity } q^2$$

$$= \int_{-h}^{\eta} \rho\left[c + \frac{acn}{\sinh mh}\cdot\cosh m(y+h)\sin(mx-nt) - gy\right]$$
$$\times\left[\frac{acn}{\sinh mh}\cdot\cosh m(y+h)\sin(mx-nt)\right]dy$$

$$= \int_{-h}^{\eta}\left[\rho(c-gy)\frac{acm}{\sinh mh}\cdot\cosh m(y+h)\sin(mx-nt)\right.$$
$$\left. + \rho\frac{a^2c^2mn}{\sinh^2 mh}\cdot\cosh^2 m(y+h)\sin^2(mx-nt)\right]dy$$

$$= \alpha \sin(mx-nt) + \beta \sin^2(mx-nt) \longrightarrow (iv)$$

where $\alpha = \int_{-h}^{\eta} \rho(c+gy) \frac{acm}{\sinh mh} \cdot \cosh m(y+h)\, dy$

$$\beta = \rho \frac{a^2 c^2 mn}{\sinh^2 mh} \int_{-h}^{\eta} \cosh^2 m(y+h)\, dy \longrightarrow (v)$$

Now, the avarage value of $\sin(mx-nt)$ and $\sin^2(mx-nt)$ over η periods are 0 and $\frac{1}{2}$.

Using this in (iv), we have the avarage rate of transmission of energy $= \alpha . 0 + \beta . \frac{1}{2} = \frac{1}{2}\beta \longrightarrow (vi)$

Again, we have

$$c^2 = \frac{g}{m} \tanh mh$$

for waves on the surface of water of depth h.

Now, $\beta = \frac{\rho}{2} a^2 \frac{g}{m} \tanh mh . \frac{mn}{\sinh^2 mh} \int_{-h}^{\eta} \left[1 + \cosh 2m(y+h)\right] dy$.

$$= \frac{1}{2} \rho a^2 g \frac{n}{\sinh mh \cosh mh} \left[(\eta+h) + \frac{1}{2m}\left\{\sinh 2m(\eta+h) - 0\right\}\right]$$

$$= \rho a^2 g n \frac{n}{\sinh 2mh} \left[h + \frac{1}{2m} \sinh mh\right]$$

as $a^2\eta$ contains a^3 and hence neglected, also $\lambda << h$.

$$= \rho a^2 g n \left[\frac{h}{\sinh 2mh} + \frac{1}{2m}\right]$$

$$= \rho a^2 g c m \left[\frac{h}{\sinh 2mh} + \frac{1}{2m}\right] ; \because c = \frac{n}{m}$$

$$= \frac{1}{2}\rho a^2 g c\left[\frac{2mh}{\sinh 2mh} + 1\right]$$

$= \rho a^2 g c_g$, where c_g is the group velocity and

$$c_g = \frac{c}{2}\left[1 + \frac{2mh}{\sinh 2mh}\right]$$

$\therefore$ W = avarage rate of transmission of energy

$$= \frac{1}{2}\beta$$

$$= \frac{1}{2}\rho a^2 g c_g$$

$= E c_g$, where $E = \frac{1}{2}\rho a^2 g$ is the total energy of a wave profile $[\eta = a\sin(mx - nt)]$ per unit length h.

This prove that the energy is transmitted at a rate equal to group velocity.

This is known as Dynamical significance of group velocity.

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