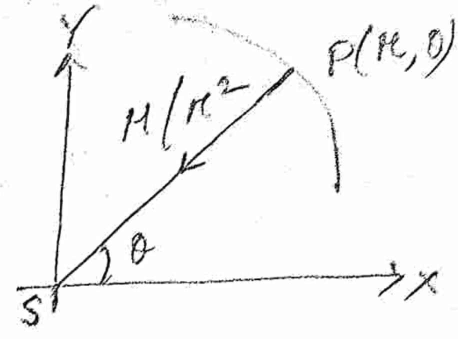
Differential eqn of the orbit of a planet:



Let SX and SY be the perpendicular axes on the plane of the orbit. Let P be the position of the planet at time t with (r, θ) as the polar co-ordinates referred to S as the pole and SX as the initial line.

The eqn of motion of the planet are

$$\ddot{r} - r\dot{\theta}^2 = -\mu/r^2 \longrightarrow (1)$$

$$\frac{1}{r}\cdot\frac{d}{dt}(r^2\dot{\theta}) = 0 \longrightarrow (2)$$

(2) gives $r^2\dot{\theta}$ = a constant = h (say) $\rightarrow (3)$

Let $r = \frac{1}{u}$ then (3) becomes

$$\dot{\theta} = \frac{h}{r^2} = hu^2$$

$$\therefore \dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\cdot\frac{du}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2} \cdot \frac{du}{d\theta} \, hu^2$$

$$= -h \cdot \frac{du}{d\theta}$$

and $\ddot{r} = \frac{d}{dt}\dot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$

$$= -h\frac{d^2u}{d\theta^2}\dot{\theta}$$

$$= -h\frac{d^2u}{d\theta^2}hu^2$$

$$= -h^2u^2\frac{d^2u}{d\theta^2}$$

Then eq^n (1) becomes

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4 = -\mu u^2$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \longrightarrow (4)$$

which is the differential eq^n of the orbit of the planet (or satellite).

$$(4) \Rightarrow 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta} = \frac{2\mu}{h^2}\frac{du}{d\theta}$$ [multiplying both sides by $2\frac{du}{d\theta}$]

$$\Rightarrow \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2 + \frac{d}{d\theta}(u^2) = \frac{2\mu}{h^2}\cdot\frac{du}{d\theta}$$

Integrating, we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2\mu u}{h^2} + c$$, where c is a constant of ~~ing~~ integration $\longrightarrow (5)$

Since $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$

$$\therefore v^2 = h^2\left(\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}\,h^2u^4$$

$$= h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$$

$$= h^2\left[\frac{2\mu u}{h^2} + c\right] \quad \text{using (5)}$$

$$= 2\mu u + ch^2 \longrightarrow (6)$$

Again $T + V = E$

$$\Rightarrow \frac{1}{2}mv^2 + \left(-\frac{\mu}{r}\right) = E \qquad \left[\because V = -\frac{GMm}{r},\ \mu = GMm\right]$$

$$\Rightarrow \frac{1}{2}v^2 - \mu u = E \quad (\text{taking } m = 1)$$

$$\Rightarrow v^2 = 2\mu u + 2E \longrightarrow (7)$$

Comparing (6) and (7), We get

$$2E = ch^2$$

Also, the ~~eqn~~ eqn of the orbit can be written as $r = \frac{l}{1+e\cos\theta}$

$$\therefore \frac{1}{r} = \frac{1+e\cos\theta}{l}$$

$$\Rightarrow u = \frac{1+e\cos\theta}{l}$$

$$\Rightarrow \frac{du}{d\theta} = -\frac{e}{l}\sin\theta$$

Since $v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$

$$2\mu u + 2E = h^2\left[\frac{e^2}{l^2}\sin^2\theta + \frac{1}{l^2} + \frac{2e\cos\theta}{l^2} + \frac{e^2}{l^2}\cos^2\theta\right] \text{ using (7)}$$

$$= \frac{h^2}{l^2}\left(e^2 + 1 + 2e\cos\theta\right)$$

$$= \frac{h^2}{l^2}\left(e^2 + 1 + 2(lu - 1)\right]$$

$$= \frac{h^2}{l^2}\left[(e^2 - 1) + 2lu\right]$$

$\because u = \frac{1+e\cos\theta}{l}$

$\Rightarrow e\cos\theta = lu - 1$

$$= \frac{\mu l}{l^2}\left[(e^2-1)+2lu\right] \quad [\because h^2 = \mu l]$$

$$= \frac{\mu}{l}(e^2-1) + 2\mu u$$

$$\Rightarrow 2E = \frac{\mu}{l}(e^2-1) \longrightarrow (8)$$

Now (7) gives

$$v^2 - 2\mu u = 2E = \frac{\mu}{l}(e^2-1) \rightarrow (9)$$

Let us suppose that the satellite is ~~prog~~ projected with velocity v_0 from a point at a distance R from the centre of force, ie ~~$u = v$~~ $v = v_0$ when $r = R = \frac{1}{u}$

$$(9) \Rightarrow V_0^2 - \frac{2\mu}{R} = 2E = \frac{\mu}{l}(e^2-1) \rightarrow (10)$$

Then the path (orbit) of the satellite is (i) a hyperbola if $e > 1$ ie if $E > 0$ or if $V_0^2 > \frac{2\mu}{R}$.

(ii) an ellipse if $e<1$ ie if $E<0$ or if $v_0^2 < \frac{2\mu}{R}$

(iii) a parabola if $e=1$ ie if $E=0$ or if $v_0^2 = \frac{2\mu}{R}$

$\therefore$ The nature of the orbit of the satellite depends on the initial velocity or the velocity projection.

<u>Velocity of a planet in its orbit.</u>

Let v be the velocity of a planet at a distance r from the sun then we have

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$$

$$\Rightarrow v^2 - 2\mu u = \frac{\mu}{l}(e^2-1) = \frac{\mu(e^2-1)}{a(1-e^2)}$$

$$\Rightarrow v^2 - \frac{2\mu}{r} = -\frac{\mu}{a}$$

$$\Rightarrow v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

for an ellipse

$$l = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2)$$