Material Energy Tensor / Energy Momentum Tensor :→

If ρ_0 is the proper density of matter and $\frac{dx^\mu}{ds}$ refers to the motion of matter, then the relativistic units (i.e. $c = G = 1$) the material energy tensor is defined as

$$T^{\mu\nu} = \rho_0 \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \longrightarrow (1)$$

In Galilian coordinate system, we have

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \quad (\because c=1)$$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = 1 - \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}.$$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = 1 - v^2 \quad \text{Taking } v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$\longrightarrow (2)$$

If ρ is the coordinate density of matter moving with velocity v relative to Galilian coordinate, then

$$\rho = \frac{\rho_0}{1 - \frac{v^2}{c^2}} = \frac{\rho_0}{1 - v^2} \quad \text{in relativistic unit.}$$

$$\therefore \quad \rho_0 = \rho(1 - v^2) = \rho\left(\frac{ds}{dt}\right)^2 \longrightarrow [\text{using } (2)$$

Hence in Galilian coordinates (1) become

$$T^{\mu\nu} = \rho\left(\frac{ds}{dt}\right)^2 \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$

$$\Rightarrow T^{\mu\nu} = \rho \frac{dx^\mu}{dt} \frac{dx^\mu}{dt} \longrightarrow (3)$$

If we write $\frac{dx^1}{dt} = u$, $\frac{dx^2}{dt} = v$, $\frac{dx^3}{dt} = w$

Then (1) becomes

$$T^{\mu\nu} = \begin{bmatrix} \rho u^2 & \rho uv & \rho uw & \rho u \\ \rho vu & \rho v^2 & \rho vw & \rho v \\ \rho wu & \rho wv & \rho w^2 & \rho w \\ \rho u & \rho v & \rho w & \rho \end{bmatrix} \longrightarrow (4)$$

In atomically constituted matter, a volume which is regarded

Now we obtain the equation

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0 \longrightarrow (6)$$

Taking first $\mu = 4$ and using (5) we get

$$\frac{\partial T^{41}}{\partial x^1} + \frac{\partial T^{42}}{\partial x^2} + \frac{\partial T^{43}}{\partial x^3} + \frac{\partial T^{44}}{\partial x^4} = 0$$

$$\Rightarrow \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \longrightarrow (7)$$

which represents the equation of continuity in hydrodynamics.

Now taking $\mu = 1$ and using (5) the equation (6) becomes

$$\frac{\partial T^{11}}{\partial x^1} + \frac{\partial T^{12}}{\partial x^2} + \frac{\partial T^{13}}{\partial x^3} + \frac{\partial T^{14}}{\partial x^4} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(P_{xx} + \rho u^2) + \frac{\partial}{\partial y}(P_{xy} + \rho uv) + \frac{\partial}{\partial z}(P_{xz} + \rho uw) + \frac{\partial}{\partial t}(\rho u) = 0$$

$$\Rightarrow \frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} + \frac{\partial P_{xz}}{\partial z} = -\left[\frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} + \frac{\partial(\rho u)}{\partial t}\right]$$

$$= -\left[u\frac{\partial(\rho u)}{\partial x} + \rho u\frac{\partial u}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + \rho v\frac{\partial u}{\partial y} + u\frac{\partial(\rho w)}{\partial z} + \rho w\frac{\partial u}{\partial z} + u\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial t}\right]$$

$$= -u\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} + \frac{\partial \rho}{\partial t}\right] - \rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right]$$