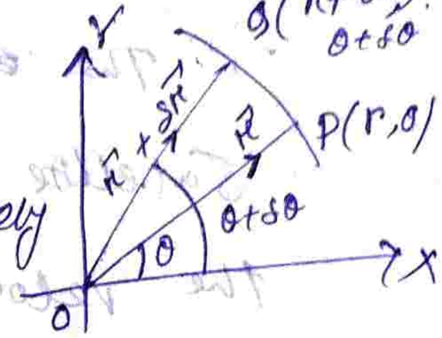
Areal Velocity and Time period of a planet (or satellite) in its orbit:

Let P and Q be the positions of a planet at time t and $t+\delta t$ respectively with p.v.s $\vec{r}$ and $\vec{r}+\delta\vec{r}$.



Also let P and Q makes an angle θ and $\theta+\delta\theta$ respectively with the initial line OX.

Now δA = area $POQ = \frac{1}{2}|\vec{OP}\times\vec{OQ}|$

$$= \frac{1}{2}|\vec{r}\times(\vec{r}+\delta\vec{r})|$$

$$= \frac{1}{2}|\vec{r}\times\delta\vec{r}|$$

$$\Rightarrow \frac{\delta A}{\delta t} = \frac{1}{2}\left|\vec{r}\times\frac{\delta\vec{r}}{\delta t}\right|$$

$$\Rightarrow \lim_{\delta t\to 0}\frac{\delta A}{\delta t} = \frac{1}{2}\left|\vec{r}\times\lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t}\right|$$

$$\Rightarrow \frac{1}{2}|\vec{r}\times\vec{v}|$$

$$= \frac{1}{2}|\vec{h}| = \frac{h}{2} = \text{constant}$$

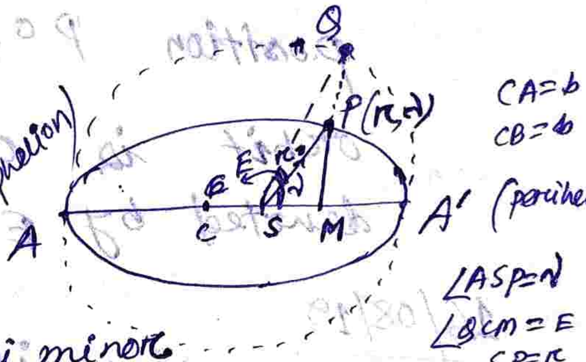
The period is given by

$$T = \frac{\pi ab}{\delta A} = \frac{\pi ab}{h/2} = \frac{2\pi ab}{\sqrt{\mu l}}$$

$$= \frac{2\pi ab}{\sqrt{\mu \frac{b^2}{a}}} \qquad \left[\begin{array}{l} \because h^2 = \mu l \\ \because l = \frac{b^2}{a} \end{array}\right]$$

$$= \frac{2\pi}{\sqrt{\mu}}\, a^{3/2}$$

Some definitions



CA=b
CB=b
∠ASP=θ
∠QCM=E
SP=r
CS=ae

Let the planet P describes an ellipse APA' with centre C, and CA, CB are the semi major and semi minor axis respectively. The sun is at its focus S. AQA' be the auxiliary circle. The co-ordinate of P referred to C as origin be $(a\cos E, b\sin E)$, Then the co-ordinate of P referred to S as origin are $x = SM = CM - CS = a\cos E - ae$

and $y = PM = b\sin E$

$PM : QM = b : a$

$\sin E = \frac{QM}{a}$

The Angle which the Radius vector SP makes with the line CA from the sun towards the perihelion is called true anomaly. In the figure $\angle ASP = \nu$ = true Anomaly.

$\sin E = $

$\frac{PM}{QM} = \frac{b}{a}$

$\Rightarrow \frac{PM}{b} = \frac{QM}{a}$

$\sin E = \frac{QM}{a}$

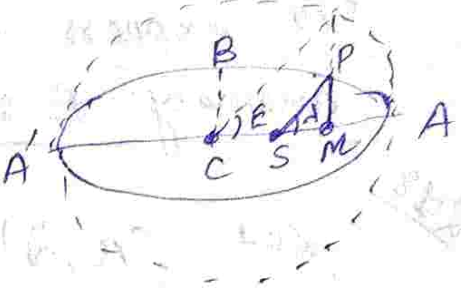
$= \frac{PM}{b} \cdot \frac{a}{a}$

$\Rightarrow PM = b \sin E$

The eccentric angle $\angle QCM$ of any position position P of the planet in its orbit is called eccentric anomaly it is denoted by E i.e. $\angle QCM = E$.

Now, if T be the periodic time of the planet, then $\frac{2\pi}{T}$ is called the mean angular velocity and is generally denoted by n and so $n = \frac{2\pi}{T}$. If P be the position of the planet at time t and τ the time at which the planet passed through perihelion then $n(t-\tau)$ is mean anomaly.

The distance of the planet from the Sun is called the heliocentric distance. The mean distance of the planet from the sun is the semi major axis of the orbit.



The semi major axis of the Earths orbit is taken as the unit of distance, and is the astronomical unit of distance. The year, equal to about $365\frac{1}{4}$ days, is often taken as unit of time.

The polar eqn of the ellipse referred to focus as pole is $\frac{l}{r} = 1 + e\cos\theta$

where $CS = ae$, $b^2 = a^2(1-e^2)$

and latus rectum $l = \frac{b^2}{a} = a(1-e^2)$

Here the angle $\angle ASP$, ie θ is denoted by v

Here the eqn of the planet orbit is

$$\frac{l}{r} = 1 + e\cos v$$

ie $\frac{a(1-e^2)}{r} = 1 + e\cos v$, where v is the true anomaly