Gravitational Field Equations (Einstein's field Equations):

Einstein wrote down (did not derive) the field equations on the basis of certain considerations as listed below:

(i) Poisson's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 4\pi G \rho \longrightarrow (1)$$

in the Newton's theory links up the gravitational potential ψ with the density of matter ρ, G being the Gravitational constant. While discussing the principle of equivalence. We note that ψ can be interpreted either as potential function or metric tensor $g_{\mu\nu}$. In order to get an analogue of the equation (1) in general theory of relativity ψ must be replaced by the metric tensor $g_{\mu\nu}$. That is to say, we consider $g_{\mu\nu}$ to be gravitational potential.

(ii) Since equation (1) does not contain higher derivatives of the Newtonian potential ψ than the second order, its relativistic analogue should involve a tensor not containing derivatives of $g_{\mu\nu}$ higher than second order.

(iii) In order to satisfy the principle of covariance, the required equation

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must be in a covariant form. This principle ensures that the field equations must have the same form in all coordinate systems. Hence they must be tensor equations.

(iv) The conservation law $T^{\mu\nu}_{;\nu} = 0$ in a curved space-time is in argument with the principle of equivalence which requires that locally in a frame in which $\Gamma^{\mu} = 0$ the flat space-time (special relativistic), the conservation law should be recovered.

All these considerations in a curved space is represented by the Riemann tensor.

$$R^{\alpha}_{\beta\nu\delta} = \{\beta\delta, \mu\}\{\mu\nu, \alpha\} - \{\beta\nu, \mu\}\{\mu\nu, \alpha\} + \frac{\partial}{\partial x^{\nu}}\{\beta\delta, \alpha\} - \frac{\partial}{\partial x^{\delta}}\{\beta\nu, \alpha\} \rightarrow (2)$$

Space-time become flat if $R^{\alpha}_{\beta\nu\delta} = 0$.

Hence the equation obtained by setting $R^{\alpha}_{\beta\nu\delta}$ equal to zero would eliminate the very possibility of a gravitational field. From the Riemann tensor we get another tensor called Ricci tensor $\alpha = \delta$ in equation (2) and summing over α

$$R_{\beta\nu} = \sum_{\alpha=1}^{4} R^{\alpha}_{\beta\nu\alpha} = \sum_{\alpha=1}^{4} \Big[\{\beta\alpha, \mu\}\{\mu\nu, \alpha\} - \{\beta\nu, \mu\}\{\mu\nu, \alpha\} + \frac{\partial}{\partial x^{\nu}}\{\beta\delta, \alpha\} - \frac{\partial}{\partial x^{\delta}}\{\beta\nu, \alpha\}\Big] \rightarrow (3)$$

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