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Ex. A canal of infinite length and rectangular section is of uniform depth h and breadth b in one part but changes gradually to depth h' ~~but~~ and breadth b' in another part. An infinite train of simple harmonic waves travelling in one direction only is propagated along the canal. Prove that if a, a' are the heights and $\frac{2\pi}{m}, \frac{2\pi}{m'}$ the lengths of the waves in the two uniform portions then

$$m \tanh mh = m' \tanh m'h'$$

and $a^2 b \sinh^2 mh (\sinh 2mh + 2mh)$

$$= a'^2 b' \sinh^2 m'h' (\sinh 2m'h' + 2m'h').$$

Sol. Let the mean profiles of the waves in the two parts be

$$\eta = a \sin(mx - nt) \text{ and } \eta' = a' \sin(m'x - n't)$$

So that $c^2 = \frac{g}{m} \tanh mh$, $c'^2 = \frac{g}{m'} \tanh m'h' \longrightarrow$ (I)

where c and c' are the wave velocities. The period of simple harmonic wave must remain the same all along the canal. This gives

$$\frac{2\pi}{n} = \frac{2\pi}{n'} \Rightarrow \frac{1}{n} = \frac{1}{n'}$$

$$\Rightarrow \frac{m}{n} \frac{1}{m} = \frac{m'}{n'} \cdot \frac{1}{m'}$$

$$\Rightarrow \frac{1}{cm} = \frac{1}{c'm'}$$

$$\Rightarrow \frac{c}{c'} = \frac{m'}{m} \longrightarrow \text{(II)}$$

From (I) and (II),

$$\frac{\frac{g}{m} \tanh mh}{\frac{g}{m'} \tanh m'h'} = \frac{m'^2}{m^2}$$

$$\Rightarrow \frac{\tanh mh}{\tanh m'h'} = \frac{m'}{m}$$

$$\Rightarrow m \tanh mh = m' \tanh m'h'.$$

Apart from the invariant periodic time, the energy transmitted will also be same in other part of the canal. Thus

$$\frac{1}{2}\frac{\rho}{a^2} g bc\left(1+\frac{2mh}{\sinh 2mh}\right)=\frac{1}{2}\cdot\frac{\rho}{a'^2} g b'c'\left(1+\frac{2m'h'}{\sinh 2m'h'}\right)$$

$$\Rightarrow a^2 bc\left(\frac{\sinh 2mh+2mh}{\sinh 2mh}\right)=a'^2 b'c'\left(\frac{\sinh 2m'h'+2m'h'}{\sinh 2m'h'}\right)$$

$$\Rightarrow \frac{a^2 bc^2 m}{cm}\left(\qquad\right)=\frac{a'^2 b c'^2 m'}{c'm'}\left(\qquad\right)$$

$$\Rightarrow \frac{a^2 bg\tanh mh}{cm}\left(\qquad\right)=\frac{a'^2 b' g\tanh mh'}{c'm'}\left(\qquad\right)$$

$$\Rightarrow \frac{a^2 b}{cm}\,\frac{\sinh mh}{\cosh mh}\left(\frac{\sinh 2mh+2mh}{2\sinh mh\cosh mh}\right)=\frac{a'^2 b}{c'm'}\operatorname{sech}^2 m'h'\left(\sinh 2m'h'+2m'h'\right)$$

$$\Rightarrow \frac{a^2 b}{cm}\cdot\operatorname{sech}^2 mh\left(\sinh 2mh+2mh\right)=\frac{a'^2 b'}{c'm'}\operatorname{sech}^2 m'h'\left(\sinh 2m'h'+2m'h'\right)$$

$$\Rightarrow a^2 b\operatorname{sech}^2 mh\left(\sinh 2mh+2mh\right)=a'^2 b'\operatorname{sech}^2 m'h'\left(\sinh 2m'h'+2m'h'\right).$$

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§: Surface tension and capillary waves:

The common surface (or interface) of two fluids which do not mix, behaves as if it were in a state of uniform tension is called surface tension. Usually surface tension is denoted by T.

Waves of the kind in which surface tension is taken into account are known as capillary waves.

Surface tension depends on temperature and also on the nature of two fluids in contact. From the theory of hydrostatics, we have the relation between surface tension and pressure as

$$p_1 - p_2 = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \longrightarrow (1)$$

where p_1, p_2 are pressures on the either side of the surface and r_1, r_2 are the principal radii of curvature of the surface. For a two dimensional wave $r_2 \longrightarrow \infty$ and η be the elevation then

$$\frac{1}{r_1} = \frac{1}{r} = -\frac{\partial^2 \eta}{\partial x^2} \longrightarrow (11)$$

by neglecting small quantities, since the slope is assumed to be very small. It follows that the above equation (1) may be expressed as,

$$p_1 - p_2 = -T\frac{\partial^2 \eta}{\partial x^2} \longrightarrow (111)$$

Boundary condition for capillary waves:

By pressure equation, we have

$$\frac{p}{\rho} + \frac{1}{2}q^2 + g\eta - \frac{\partial \phi}{\partial t} = c$$

for gravity controlled waves. If p_1 and p_2 be the pressure inside and outside of a wave surface, then neglecting the quantity q^2 we have

$$\frac{p_1}{\rho} + g\eta - \frac{\partial \phi}{\partial t} = c$$

or, $p_1 = \rho\left(\frac{\partial \phi}{\partial t} - g\eta\right) + c$

So that, $p_1 - p_2 = \rho\left(\frac{\partial \phi}{\partial t} - g\eta\right) + c - p_2$

or, $p - p_2 = \rho\left(\frac{\partial \phi}{\partial t} - g\eta\right)$, adjusting ϕ suitably.

Using this in (III)

$$\rho\left(\frac{\partial \phi}{\partial t} - g\eta\right) = -T\frac{\partial^2 \eta}{\partial x^2}$$

Differentiating partially w.r.to 't', we have

$$\rho\left(\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \eta}{\partial t}\right) = -T\frac{\partial}{\partial t}\left(\frac{\partial^2 \eta}{\partial x^2}\right)$$

$$= -T\frac{\partial^2}{\partial x^2}\left(\frac{\partial \eta}{\partial t}\right) \longrightarrow \text{(IV)}$$

Also, the kinematical surface condition is,

$$\frac{\partial \eta}{\partial t} = \frac{\partial \psi}{\partial x}$$

$\therefore$ (IV) becomes,

$$\psi\left(\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \psi}{\partial x}\right) = -T\frac{\partial^2}{\partial x^2}\left(\frac{\partial \psi}{\partial x}\right)$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \psi}{\partial x} + \frac{T}{\rho}\frac{\partial^3 \psi}{\partial x^3} = 0$$

This is called capillary surface condition.

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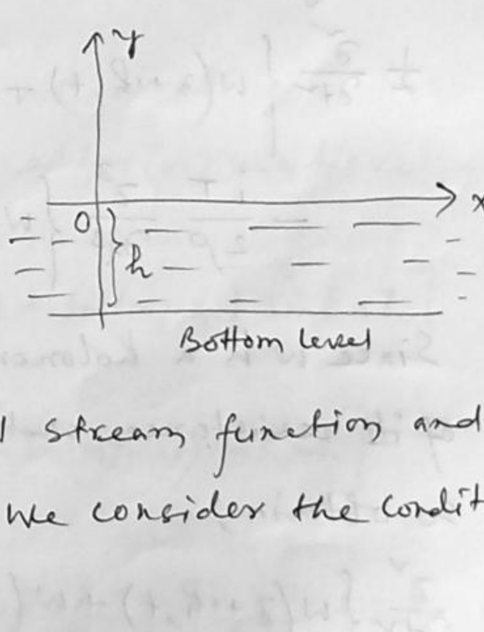
Ex. Define surface tension and capillary waves. Show that the free surface condition is

$$\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \psi}{\partial x} + \frac{T}{\rho}\frac{\partial^3 \psi}{\partial x^3} = 0$$

where ϕ, ψ are velocity potential and stream function respectively for two-dimensional motion and T is the surface tension.

§. Cisotti's equation and effects of capillarity:

Let us choose the axes as shown in the figure, the capillary surface condition is



$$\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \psi}{\partial x} + \frac{T}{\rho}\frac{\partial^3 \psi}{\partial x^3} = 0 \longrightarrow (1)$$

where ϕ, ψ & T are velocity potential stream function and surface tension respectively. Again we consider the condition

$$\psi = 0 \text{ at } y = 0 \longrightarrow (II)$$

as accross the bottom there is no flow.

Also, the complex potential is given by

$$w = w(z,t) = \phi(x,y,t) + i\psi(x,y,t) \longrightarrow (III)$$

condition (III) therefore states that w is real when $y=0$ so that by the principle of analytic condition the holonomic function

W can be continued into the region for which y is negative more precisely $-h \le y \le 0$.

Now, $w(\bar{z},t) = \phi(x,y,t) - i\psi(x,y,t) \longrightarrow$ (IV)

$\therefore$ By (III) and (IV),

$$\phi(x,y,t) = \frac{1}{2}\left[w(z,t) + w(\bar{z},t)\right]$$

and $$\psi(x,y,t) = \frac{1}{2} i \left[w(z,t) - w(\bar{z},t)\right]$$

which implies

$$\phi(x,y,t) = \frac{1}{2}\left[w(x+iy,t) + w(x-iy,t)\right]$$

and $$\psi(x,y,t) = \frac{1}{2} i \left[w(x+iy,t) - w(x-iy,t)\right]$$

Putting $y = h$ and substituting in (I), we get,

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\left\{w(x+ih,t) + w(x-ih,t)\right\} + ig\frac{1}{2}\frac{\partial}{\partial x}\left\{w(x+ih,t) - w(x-ih,t)\right\}$$
$$-\frac{iT}{2\rho}\frac{\partial^3}{\partial x^3}\left\{w(x+ih,t) - w(x-ih,t)\right\} = 0 \longrightarrow \text{(V)}$$

Since w is a holonomic function, (V) holds for any point in the region of its existance and we can therefore write z for x, and so obtain,

$$\frac{\partial^2}{\partial t^2}\left\{w(z+ih,t) + w(z-ih,t)\right\} + ig\frac{\partial}{\partial z}\left\{w(z+ih,t) - w(z-ih,t)\right\}$$
$$-i\frac{T}{\rho}\frac{\partial^3}{\partial z^3}\left\{w(z+ih,t) - w(z-ih,t)\right\} = 0$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}(w_1 + w_2) + ig\frac{\partial}{\partial z}(w_1 - w_2) = i\frac{T}{\rho}\frac{\partial^3}{\partial z^3}(w_1 - w_2), \longrightarrow \text{(VI)}$$

where $w_1 = w(z+ih,t)$,
$w_2 = w(z-ih,t)$.

This equation (vi) was given by Cisotti and is known as Cisotti's equation.

Effect of capillarity on surface waves:

Let $W(z,t) = A\cos(mz - nt)$ be the complex potential of the surface waves in water of depth h.

$$\therefore W_1 - W_2 = W(z+ih,t) - W(z-ih,t)$$

$$= A\cos[m(z+ih)-nt] - A\cos[m(z-ih)-nt]$$

$$= -2Ai\sin(mz-nt)\sinh mh$$

$||^{ly}$

$$W_1 + W_2 = W(z+ih,t) + W(z-ih,t)$$

$$= A\cos[m(z+ih)-nt] + A\cos[m(z-ih)-nt]$$

$$= 2A\cos(mz-nt)\cosh mh$$

Substituting these in Cisotti's equation (vi) we have,

$$\frac{\partial^2}{\partial t^2}\left[2A\cos(mz-nt)\cosh mh\right] + ig\frac{\partial}{\partial z}\left[-2Ai\sin(mz-nt)\sinh mh\right]$$

$$= i\frac{T}{\rho}\frac{\partial^3}{\partial z^3}\left[-2Ai\sin(mz-nt)\sinh mh\right]$$

$$\Rightarrow -n^2\cos(mz-nt)\cosh mh + mg\cos(mz-nt)\sinh mh$$

$$= -\frac{T}{\rho}m^3\cos(mz-nt)\sinh mh.$$

$$\Rightarrow n^2\cosh mh = mg\sinh mh + \frac{T}{\rho}m^3\sinh mh = m\left(g + \frac{T}{\rho}m^2\right)\sinh mh$$

$$\Rightarrow n^2 = m\left(g + \frac{T}{\rho}\cdot m^2\right)\tanh mh$$

$$\Rightarrow \frac{n^2}{m^2} = \left(\frac{g}{m} + \frac{T}{\rho} m\right) \tanh mh$$

$$\Rightarrow c^2 = \left(\frac{g}{m} + \frac{T}{\rho} m\right) \tanh mh$$

which gives the velocity of propagation of waves of length $\frac{2\pi}{m}$

When h is large compared to the wave length λ (i.e. for deep sea, when $h \to \infty$) then $\tanh mh \to 1$.

$$\therefore c^2 = \frac{g}{m} + \frac{T}{\rho} m$$

$$= \left(\frac{g\lambda}{2\pi} + \frac{T}{\rho} \frac{2\pi}{\lambda}\right) \quad ; \lambda = \frac{2\pi}{m}$$

$$\Rightarrow c = \left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda}\right)^{\frac{1}{2}}.$$

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Ex. Show that the velocity of propagation of a wave of small wave length λ on the surface of an inviscid incompressible fluid of infinite depth is given by

$$c = \left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda}\right)^{\frac{1}{2}}$$

where g, ρ and T denote the acceleration due to gravity, density and the surface tension of the fluid respectively.