To express true anomaly v interms of eccentric anomaly E:

2018

Let (r, v) be polar co-ordinates of any position of the planet P referred to S as pole whose cartesian co-ordinate is (x, y)

Then $x = a\cos E - ae$, $y = b\sin E \longrightarrow (1)$

and $x = r\cos v$, $y = r\sin v \longrightarrow (2)$

Then from (1) and (2) we have

$$r\cos v = a\cos E - ae = a(\cos E - e) \rightarrow (3)$$

and $r\sin v = b\sin E \longrightarrow (4)$

squaring and adding (3) and (4), we get

$$r^2 = a^2\left[\cos^2 E + e^2 - 2e\cos E + (1-e^2)\sin^2 E\right]$$

$$\left[\because b^2 = a(1-e^2)\right]$$

$$= a^2\left[1 - 2e\cos E + e^2\cos^2 E\right]$$

$$= a^2(1 - e\cos E)^2$$

$$\Rightarrow r = a(1 - e\cos E) \longrightarrow (5)$$

Now $2r\sin^2\frac{v}{2} = r(1-\cos v)$

$$= a(1 - e\cos E)$$

$$= a\{(1+e) - \cos E(1+e)\}$$

$$= a(1+e)(1-\cos E) \longrightarrow ⑥$$

Similarly $\quad 2r\cos^2\frac{\nu}{2} = r(1+\cos\nu)$

$$= a(1-e\cos E) + a(\cos E - e)$$

$$= a[(1-e) + \cos E(1-e)]$$

$$= a(1-e)(1+\cos E) \longrightarrow ⑦$$

dividing ⑥ by ⑦ we get

$$\tan^2\frac{\nu}{2} = \frac{(1+e)(1-\cos E)}{(1-e)(1+\cos E)}$$

$$\Rightarrow \tan\frac{\nu}{2} = \left(\frac{1-e}{1-e}\right)^{\frac{1}{2}} \tan\frac{E}{2} \longrightarrow ⑧$$

From ⑧, we get

$$\frac{\tan\frac{\nu}{2}}{\tan\frac{\nu}{2}} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}$$

applying components and dividends, we find

$$\frac{\tan\frac{\nu}{2} - \tan\frac{E}{2}}{\tan\frac{\nu}{2} + \tan\frac{E}{2}} = \frac{\sqrt{1+e} - \sqrt{1-e}}{\sqrt{1+e} + \sqrt{1-e}}$$

$$\Rightarrow \frac{\sin\left(\frac{\nu - E}{2}\right)}{\sin\left(\frac{\nu + E}{2}\right)} = \frac{\left(\sqrt{1+e} - \sqrt{1-e}\right)^2}{(1+e) - (1-e)}$$

$$= \frac{1+e(1-e) - 2\left(\sqrt{1-e^2}\right)}{2e}$$

$$= \frac{1-\sqrt{1-e^2}}{e}$$

$$\Rightarrow \sin\left(\frac{\nu - E}{2}\right) = \left\{\frac{1-\sqrt{1-e^2}}{e}\right\} \sin\left(\frac{\nu - E}{2} + E\right) \longrightarrow (9)$$

Now (9) is of the form $\sin x = n \sin(x + \alpha)$

and if $n < 1$, then

$$x = n\sin\alpha + \frac{n^2}{2}\sin 2\alpha + \frac{n^3}{3}\sin 3\alpha + \cdots \longrightarrow (10)$$

Here, $n = \dfrac{1-\sqrt{1-e^2}}{e}$

$$= \frac{1 - \left\{1 - \frac{1}{2}e^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{L^2} e^4 + \cdots\right\}}{e}$$

$$= \frac{1}{2}e + \frac{1}{8}e^3 + \cdots$$

$$\Rightarrow n^2 = \frac{1}{4}e^2$$

$$n^3 = \frac{1}{8}e^3, \text{ upto } e^3$$

Putting the values of x, α, n, n^2, n^3 in (10),

we get $\frac{\nu - E}{2} = \left(\frac{e}{2} + \frac{e^3}{8}\right)\sin E + \frac{1}{2}\cdot\frac{1}{4}\cdot e^2 \sin 2E$

$$+ \frac{1}{3}\cdot\frac{1}{8}e^3 \sin 3E$$

$$\Rightarrow \nu = E + \left(e + \frac{1}{4}e^3\right)\sin E + \frac{1}{4}e^2 \sin 2E$$

$$+ \frac{1}{12}e^3 \sin 3E \quad (\text{retaining upto } e^3)$$

which is the required relation.

//