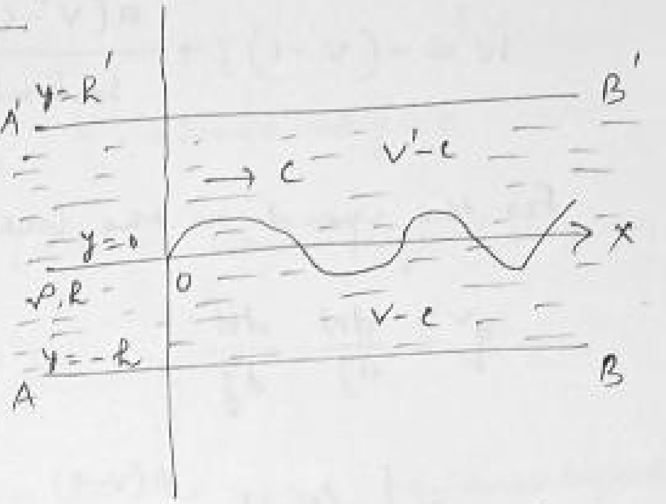
§: Capillary waves at an interface:



Let a liquid of density ρ' and depth h' move with velocity v' over another liquid of density ρ and depth h moving in the same direction with velocity v as shown in figure. Let the liquids be bounded by two fixed horizontal planes $A'B'$ and AB.

Also, let c be the velocity of propagation of oscillatory waves at the interface of two liquids, in the direction and choose the coordinates axes as shown in figure. Now, to reduce the wave profile to rest, we suppose on the whole mass, a velocity equal and opposite to that of propagation of waves. The velocities of the stream are then $v'-c$ and $v-c$ respectively.

The complex potential for the lower liquid moving with velocity $-(v-c)$ in the $-$ve direction of x-axis, is obtained by substituting $-(v-c)$ in place of c in the expression of complex potential

i.e. $$W = cz + \frac{ac}{\sinh mh}\cos m(z+ih) \longrightarrow (i)$$

Hence the complex potential for the lower liquid is

$$W = -(v-c)z - \frac{a(v-c)}{\sinh mh}\cos m(z+ih) \longrightarrow (ii)$$

Also, the complex potential W' for the upper liquid is obtained by replacing h by $-h$ and v by v' in (ii). Thus

We have,

$$w' = -(v'-c)z + \frac{a(v'-c)}{\sinh mh'} \cos m(z-ih) \longrightarrow (\text{III})$$

For the speed in the lower liquid, we have,

$$q^2 = \frac{dw}{dz} \cdot \frac{d\bar{w}}{d\bar{z}}$$

$$= \left[-(V-c) + \frac{a(V-c)}{\sinh mh} \cdot m \sin m(z+ih)\right] \times \left[-(V-c) + \frac{a(V-c)m}{\sinh mh} \sin m(\bar{z}-ih)\right]$$

$$= \left[(V-c)^2 - \frac{a(V-c)^2}{\sinh mh} m \sin m(\bar{z}-ih) - \frac{a(V-c)^2}{\sinh mh} m \sin m(z+ih)\right], (\text{approx})$$

$$= (V-c)^2 \left[1 - \frac{2ma}{\sinh mh} \sin mx \cosh m(y+h)\right] \longrightarrow (\text{IV})$$

($\because$ a is small so a^2 and higher powers of a are neglected)

For the speed in the upper liquid, we have,

$$q'^2 = \frac{dw'}{dz} \cdot \frac{d\bar{w}'}{dz}$$

$$\Rightarrow q'^2 = \left[-(v'-c) - \frac{a(v'-c)m}{\sinh mh'} \sin m(z-ih')\right] \times \left[-(v'-c) - \frac{a(v'-c)m}{\sinh mh'} \sin m(\bar{z}+ih')\right]$$

$$= \left[(v'-c)^2 + \frac{a(v'-c)m}{\sinh mh'} \sin m(\bar{z}+ih') + \frac{a(v'-c)m}{\sinh mh'} \sin m(z-ih)\right]$$

$$= (v'-c)^2 \left[1 + \frac{am}{\sinh mh'} \left\{\sin m(\bar{z}+ih') + \sin m(z-ih')\right\}\right]$$

$$= (v'-c)^2 \left[1 + \frac{2ma}{\sinh mh'} \sin mx \cosh m(y+h')\right] \longrightarrow (\text{V})$$

; $\because \cos ix = i \cosh x$.

To find the speed at the interface due to the lower liquid, we put $y=0$ in (IV) so that

$$q_0^2 = (V-c)^2\left[1-2m\eta \coth mh\right] \;;\; \because \eta = a\sin mx \text{ at } y=0.$$

11^y the speed q_0' at the interface, due to upper liquid is given by

$$q_0'^2 = (V'-c)^2\left[1+2m\eta \coth mh'\right]$$

The expressions for the pressure for the upper and lower liquids are given by

$$\frac{p'}{\rho'} + \frac{1}{2}q_0'^2 + g\eta = \text{constant}$$

$$\Rightarrow p' + \frac{1}{2}\rho' q_0'^2 + g\rho'\eta = \text{constant}$$

and $\frac{p}{\rho} + \frac{1}{2}q_0^2 + g\eta = \text{constant}$

$$\Rightarrow p + \frac{1}{2}\rho q_0^2 + g\rho\eta = \text{constant}$$

where $(p-p') + \frac{1}{2}\rho q_0^2 - \frac{1}{2}\rho' q_0'^2 + g\eta(\rho-\rho') = \text{constant}. \longrightarrow (VI)$

Since $p - p' = -T\frac{\partial^2\eta}{\partial x^2} = Tam^2\sin mx \longrightarrow (VII)$

where T is the surface tension.

from (VI) & (VII), we get,

$$Tam^2\sin mx + \frac{1}{2}\rho(V-c)^2\left[1-2ma\sin mx\coth mh\right]$$

$$-\frac{1}{2}\rho'(V'-c)^2\left[1+2ma\sin mx\coth mh'\right] + ag\sin mx(\rho-\rho') = \text{const.}$$

Equating the coefficients of $\sin mx$ on both sides, we get,

$$Tm^2 + g(\rho-\rho') = m\rho(V-c)^2\coth mh + m\rho'(V'-c)^2\coth mh' \longrightarrow (VIII)$$

Cor: If the depth of liquids be great compared to the wave lengths then $\coth mh = 1 = \coth mh'$ and further if they are at rest (save for the wave motion) then $V' = V = 0$ and also if we take $c = c_0$, then (VII) becomes,

$$Tm^2 + g(\rho - \rho') = m\rho c_0^2 + m\rho' c_0^2 = m(\rho + \rho')c_0^2$$

$$\Rightarrow c_0^2 = \frac{Tm^2 + g(\rho - \rho')}{m(\rho + \rho')}$$

$$= \frac{g}{m}\frac{(\rho - \rho')}{(\rho + \rho')} + \frac{Tm}{(\rho + \rho')}$$

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Ex. Find the velocity c_0 of propagation of waves of length $\frac{2\pi}{m}$ at the common surface of two liquids when surface tension is taken into account, and show that, if the liquids are deep compared to the wave length and are undisturbed save for the wave motion then,

$$c_0^2 = \frac{g}{m}\frac{(\rho - \rho')}{(\rho + \rho')} + \frac{Tm}{\rho + \rho'}$$

where T is the surface tension.

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Ex. Two liquids which donot mix, occupy the region between two fixed horizontal planes, the upper of density ρ' and mean depth h' is flowing with the general velocity V over lower, which is of density ρ and mean depth h, and is rest except for wave motion. Prove, neglecting viscosity, that the velocity v of waves of length $\frac{2\pi}{K}$, travelling over the common surface with the direction of U, is given by,

$$\rho v^2 \coth kh + \rho'(U-v)^2 \coth kh' = TK + g(\rho-\rho')/k$$

where T is the surface tension.

Sol^n. Previous as equation (VIII),

$$Tm^2 + g(\rho-\rho') = (V-c)^2 m\rho \coth mh' + (V'-c)^2 m\rho' \coth mh'$$

As per given question, here we replace V by 0, v' by U, m by k and c by v, then

$$TK^2 + g(\rho-\rho') = v^2 K\rho \coth mh + (U-v)^2 K \rho' \coth mh'$$

Dividing by K

$$\rho v^2 \coth kh + (U-v)^2 \rho' \coth kh' = Tk + g(\rho-\rho')/k.$$

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