Electromagnetic Induction
Lecture 5

Manoj Kr. Das
Associate Professor
Physics Department
J N College, Boko

Reciprocity Theorem:

The mutual inductance between two coils is used to evaluate the effect of changing the current in one coil on the behaviour of the current in the second coil. Let

$$E_{21} = M_{21}\left(\frac{di_1}{dt}\right) \rightarrow (i)$$

This induce emf will produce current i_2 in the secondary coil which also produce induce emf in the secondary coil. Thus

$$E_{12} = M_{12}\left(\frac{di_2}{dt}\right) \rightarrow (ii)$$

The force experienced by an element ∂l_2 of the secondary coil carrying current i_2 and assumed to be placed in a magnetic field B_1 (due to current i_1 in the primary coil) is given by

$$\partial F_1 = i_2 \partial l_2 \times B_1 \rightarrow (iii)$$

Therefore the amount of work done in displacing this element by a distance ∂r is

$$\partial W_2 = [i_2(\partial l_2 \times B_1)]\partial r \rightarrow (iv)$$

$$\Rightarrow \partial W_2 = i_2 \partial \varphi_{21} \rightarrow (v)$$

Where $\partial \varphi_{21}$ is the flux linked with the volume of parallelopiped of $(\partial l_2 \times B_1)\partial r$.

This relation is same for motion of secondary coil. Thus ∂W_2 represent the work done in displacing a secondary coil by a distance ∂r and $\partial \varphi_{21}$ is the additional flux linked with the secondary coil by virtue of current in the primary coil. The total work done in bringing the secondary coil from far apart to a point where flux linkage is φ_{21} .

$$W_2 = \int \partial W_2 = \int i_2 \partial \varphi_{21} = i_2 \varphi_{21} \rightarrow (vi)$$

Similarly the work done in bringing the primary coil keeping the secondary coil fixed. Therefore

$$W_1 = \int i_1 \partial \varphi_{12} = i_1 \varphi_{12} \rightarrow (vii)$$

Since the initial states are equivalent regardless of the way. Hence

$$W_1 = W_2$$

$$\Rightarrow i_1 \varphi_{12} = i_2 \varphi_{21}$$

$$\Rightarrow i_1(M_{12}i_2) = i_2(M_{21}i_1)$$
$$\Rightarrow \; M_{12} = M_{21} \rightarrow (viii)$$

Hence the mutual inductance of the coils remains same whether the current is flowing in first or second coil. This is known as Reciprocity Theorem of mutual inductances.

If $M_{12} = M_{21} = M$ then the amount of work done and the potential energy of the coil is

$$\text{W} = W_1 = W_2 = Mi_1i_2 \; \rightarrow (ix)$$

Energy Stored in an Inductor:

Let a resistance of value R an inductor having self inductance L and a battery of e.m.f. e are connected in series as shown in Fig1. Let I be the current flowing in the circuit at any instant t.

Therefore

$$e - L\frac{dI}{dt} = RI \rightarrow (x)$$

If dQ be the charge moved through the circuit then the work done by the e.m.f. e will be

$$dW = edQ = eIdt \rightarrow (xi)$$

$$\Longrightarrow \frac{dW}{dt} = eI = \left(RI + L\frac{dI}{dt}\right) I$$

$$\Longrightarrow \frac{dW}{dt} = RI^2 + LI\frac{dI}{dt} \rightarrow (xi)$$

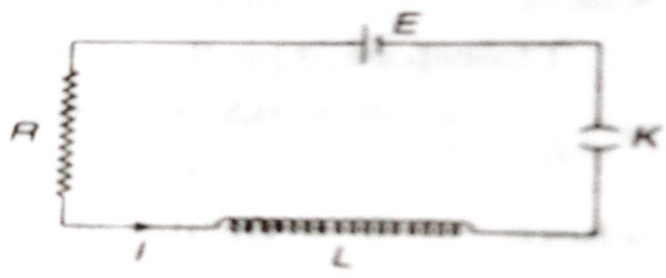

Fig 1

Therefore the total work done will be

$$W = \int \frac{dW}{dt} dt = R \int I^2 dt + L \int I \, dI$$

$$W = R \int I^2 dt + \frac{1}{2} L I^2 \rightarrow (xi)$$

The first term of the above equation represents the energy dissipated in the form of joule heat and the second term is the energy stored in the inductance.

Magnetic Energy Stored in a series of Inductance:

Let us assume that the initial current in all circuits at $t = 0$ is zero and attain their equilibrium value at time $t = T$.

The induced e.m.f. in the kth circuit will be

$$e_k = \frac{d\varphi_k}{dt} \rightarrow (xii)$$

Let the current in each circuit and the flux through it be some fraction α of the equilibrium values. Therefore

$$I_k(t) = \alpha I_k \; and \;\; \varphi_k(t) = \alpha\varphi_k \;\; \rightarrow (xiii)$$

Therefore total work done by the k[th] circuit will be

$$W_k = \int_0^T e_k I_k(t) dt \rightarrow (xiv)$$

$$W_k = \int_0^T \varphi_k \frac{d \propto}{dt} \alpha I_k dt$$

$$W_k = I_k \varphi_k \int_0^1 \alpha d\alpha$$

$$W_k = \frac{1}{2} I_k \varphi_k \rightarrow (xv)$$

Therefore for all circuit

$$W = \frac{1}{2}\sum_{k} I_k\varphi_k \rightarrow (xvi)$$

Here $$\varphi_k = L_kI_k + \sum\nolimits_{k\neq j} M_{kj}I_j \rightarrow (xvii)$$

Therefore for a pair of inductance

$$W = \frac{1}{2}L_1{I_1}^2 + \frac{1}{2}L_2{I_2}^2 + M_{12}I_1I_2 \rightarrow (xviii)$$

Magnetic Energy Density:

Amount of magnetic energy stored per unit volume is called the energy density of the magnetic field. Considering a solenoid of length l having N turns and carrying a current i_0 , magnetic induction B inside the solenoid is

$$B = \mu_0\mu_r\frac{Ni_0}{l} \rightarrow (xix)$$

Now coefficient of self induction of solenoid is given by

$$L = \mu_0 \mu_r n N A = \mu_0 \mu_r \frac{N^2 A}{l} \rightarrow (xx)$$

Energy stored in the magnetic field is

$$W = \frac{1}{2} L {i_0}^2 \rightarrow (xxi)$$

Again we know

$$i_0 = \frac{Bl}{\mu_0 \mu_r N} \rightarrow (xxii)$$

Therefore

$$W = \frac{1}{2} \frac{\mu_0 \mu_r N^2 A}{l} \times \left(\frac{Bl}{\mu_0 \mu_r N} \right)^2 \rightarrow (xxiii)$$

$$w = \frac{1}{2} \frac{B^2}{\mu_0 \mu_r} Al$$

$$w = \frac{1}{2} \frac{B^2}{\mu_0 \mu_r} V$$

Therefore energy density

$$\frac{W}{V} = \frac{1}{2} \frac{B^2}{\mu_0 \mu_r} \rightarrow (xxiv)$$

For air core solenoid $\mu_r = 1$, then energy density is $\frac{1}{2} \frac{B^2}{\mu_0}$.

Thus energy density of magnetic field varies directly as the squre of strength of magnetic field.

Current Density:

Current density at any point in a conductor is defined as current passing through an infinite small area held at that point in a direction perpendicular to the direction of flow of charge

$$dI = J.dA \rightarrow (xxv)$$

Therefore

$$I = \int dI = \iint JdA = J.A = JAcos\theta \rightarrow (xxvi)$$

Here current density is same across the cross-section. Therefore

$$J = \frac{I}{A} \rightarrow (xxvii)$$

Therefore magnitude of current density is defined as amount of electric current per unit area.

Equation of Continuity:

It is the law of conservation of charge. Let amount of current dI ,leaving an area dA is

$$dI = JdA \rightarrow (xxviii)$$

Therefore

$$I = \iint JdA \rightarrow (xxix)$$

Again we know

$$I = -\frac{dq}{dt} \rightarrow (xxx)$$

Therefore

$$\iint JdA = -\frac{dq}{dt} \rightarrow (xxxi)$$

But we know that $q = \iiint \rho dV$. Therefore

$$\iint JdA = -\iiint \frac{d\rho}{dt} dV \rightarrow (xxxii)$$

Again from Gausses Divergence Theorem we know

$$\iint JdA = \iiint \nabla . JdV \rightarrow (xxxiii)$$

So from equation $(xxxii)$ and $(xxxiii)$, we can write

$$\iiint \nabla . JdV = -\iiint \frac{d\rho}{dt} dV$$

$$\Rightarrow \iiint \left(\nabla . J + \frac{d\rho}{dt}\right) dV = 0$$

$$\Rightarrow \left(\nabla . J + \frac{d\rho}{dt}\right) = 0 \rightarrow (xxxiv)$$

This is the expression for Equation of Continuity.

Displacement Current:

Ampere's circuital law says that

$$\oint Hdl = I \rightarrow (xxxv)$$

But we know that current

$$I = \iint Jds \rightarrow (xxxvi)$$

From equation $(xxxv)$ and $(xxxvi)$ we get

$$\oint Hdl = \iint Jds \rightarrow (xxxvii)$$

But theorem says that

$$\oint Hdl = \iint \nabla \times H\, ds \rightarrow (xxxviii)$$

Comparing these two equations we get

$$\iint \nabla \times Hds = \iint Jds$$

$$\iint (\nabla \times H - J)ds = 0 \rightarrow (xxxix)$$

As the surface is arbitrary therefore integral must vanish i.e.

$$\nabla \times H - J = 0$$

$$\nabla \times H = J \rightarrow (xxxx)$$

If this equation is to be examine for time varying field, since divergence of a curl of any vector is always zero therefore then

$$\nabla.(\nabla \times H) = \nabla.J$$

$$\nabla.J = 0 \rightarrow (xxxxi)$$

Again equation of continuity says that

$$\nabla.J + \frac{d\rho}{dt} = 0$$

$$\nabla.J = -\frac{d\rho}{dt} \rightarrow (xxxxii)$$

As from this equation $\nabla.J = 0$ only if $\frac{d\rho}{dt} = 0$.

Maxwell assumed that the definition of current density (J) is incomplete and hence J_d must be added to the it. Then total current density should be $(J + J_d)$. Thus

$$\nabla \times H = J + J_d \rightarrow (xxxxiii)$$

Taking divergence of this equation we get

$$\nabla . (\nabla \times H) = \nabla (J + J_d)$$

But $\nabla . (\nabla \times H) = 0$, since divergence of any curl of any vector is always zero therefore

$$\nabla (J + J_d) = 0 \rightarrow (xxxxiv)$$

$$\nabla . J + \nabla . J_d = 0$$

$$\nabla J_d = -\nabla . J \rightarrow (xxxxv)$$

But we know $\nabla . J = -\frac{\partial \rho}{\partial T}$

Again Gauss's theorem in differential form gives

$$\nabla D = \rho \rightarrow (xxxxvi)$$

Therefore

$$\nabla J_d = \frac{\partial}{\partial t}(\nabla D)$$

$$\nabla J_d = \nabla\left(\frac{\partial D}{\partial t}\right) \rightarrow (xxxxvi)$$

This equation gives

$$J_d = \frac{\partial D}{\partial t} \rightarrow (xxxxvii)$$

Therefore modified form of Ampere's law is

$$\nabla \times H = \frac{\partial D}{\partial t} \rightarrow (xxxxvii)$$

So Maxwell added to Amper's law to include time vaying field is known as Displacement Current, because it arises when electric displacement vector D changes with time. By adding this term Maxwell assumed that this is as effective as conduction current for producing magnetic field.

Maxwell Equation:

The basic laws of electricity and magnetism are

(a)Gauss's theorem as applied to electric field

$$\nabla . D = \rho \rightarrow (xxxxviii)$$

Where $D = \varepsilon E$,where ε being the permittivity of the medium and E be the electric intensity.

(b)Gauss's theorem as applied to magnetic field

$$\nabla . B = 0 \rightarrow (xxxxix)$$

Where B being the magnetic induction

(c)Faraday's law of induction

$$\nabla \times E = -\frac{\partial B}{\partial t} \rightarrow (xxxxx)$$

(d)Ampere's Circuital theorem

$$\nabla \times H = J \rightarrow (xxxxxi)$$

Where J being the current density

Now equation $(xxxxviii)$, $(xxxxix)$ and $(xxxxx)$ are valid for both the static and dynamic fields. But Ampere's Circuital therorem is obtained from steady state observations so its validity for time varying fields have to be examined. Now taking divergence of both side of equation $(xxxxxi)$, we get

$$\nabla(\nabla \times H) = \nabla J = 0$$

The above equation is also true for steady field. Suppose the current changes with time. In this case the result is incompatible with the principle of conservation of energy i.e. the equation of continuity

$$\nabla . J + \frac{\partial \rho}{\partial t} = 0 \rightarrow (xxxxxii)$$

Maxwell said that the difficulty in case of time varying fields arose due to incomplete definition of the total current density. Now from equation $(xxxxxii)$

$$\nabla . J = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\nabla D) = \nabla\left(-\frac{\partial D}{\partial t}\right)$$

$$\nabla . \left(J + \frac{\partial D}{\partial t}\right) = 0 \rightarrow (xxxxxiii)$$

Therefore J in Ampere's Circuital Theorem must be replaced by $\left(J + \frac{\partial D}{\partial t}\right)$. Taking the above modification of circuital form of Ampere's law we can obtained

$$\nabla . H = J + \frac{\partial D}{\partial t} \rightarrow (xxxxxiv)$$

The vector is called conduction or charge transported current density and the second term arise from the variation of electric displacement with time is called displacement current density. Therefore the equations which the field vectors $E, B, D \; and \; H$ satisfy are

$$\nabla . D = \rho \rightarrow (xxxxxv)$$

$$\nabla . B = 0 \rightarrow (xxxxxvi)$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \rightarrow (xxxxxvii)$$

$$\nabla \times H = J + \frac{\partial D}{\partial t} \rightarrow (xxxxxviii)$$

The above equations $(xxxxxxv)$, $(xxxxxxvi)$, $(xxxxxxvii)$ and $(xxxxxxviii)$ are known as Maxwell's Equations for electromagnetic field.