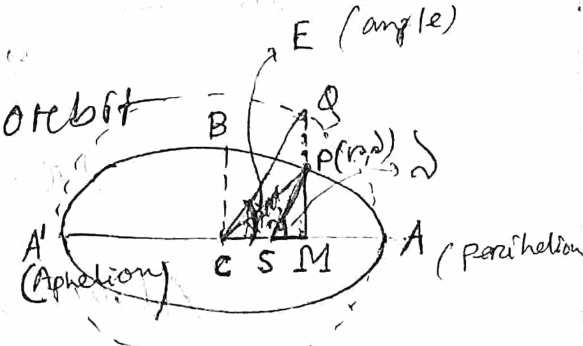
Keplers Equation

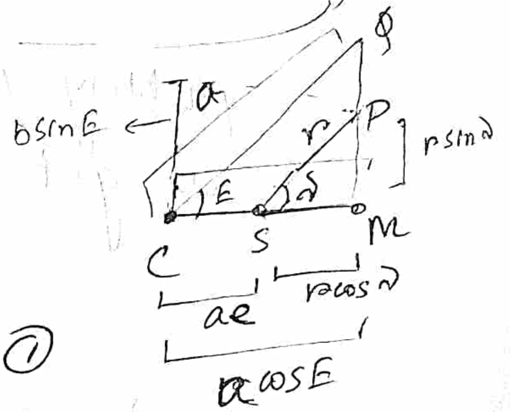
2017

Let us consider an elliptic orbit whose semi major axis is a and ~~and~~ an auxiliary circle of radius ~~of radius~~ a, that has been circumscribed about ellipse.



Let P be the position of the planet after

time "t" -
- after starting from perihelion position. Let S be the position of the Sun, also let ϑ be the true anomaly, E the Eccentric anomaly and m the mean anomaly. Then



$$\Rightarrow r\cos\vartheta = a\cos E - ae \longrightarrow (1)$$

$$r\sin\vartheta = b\sin E$$
$$= a\sqrt{1-e^2}\sin E \longrightarrow (2)$$

$[\because b^2 = a^2(1-e^2)]$

Then squaring (1) and (2), then adding, we get

$$r^2 = a^2(1-e\cos E)^2$$
$$\Rightarrow r = a(1-e\cos E) \longrightarrow (3)$$

Also, $m = n(t-\tau) = nt$
$$= \frac{2\pi t}{T} \longrightarrow (4) \quad [\tau = 0$$

where T is the periodic time

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \longrightarrow (5)$$

Now from (3), $\dot{r} = ae \sin E . \dot{E} \longrightarrow (6)$

The eqn of the orbit is given

by $\frac{l}{r} = 1 + e\cos\theta$

$$\therefore -\frac{l}{r^2} \dot{r} = - e\sin\theta \dot{\theta}$$

$$\Rightarrow \dot{r} = \frac{e\sin\theta}{l} (r^2 \dot{\theta})$$

$$= \frac{\sin\theta}{l} h \longrightarrow (7)$$

$[\because r^2\dot{\theta} = h]$

(6) and (7) gives

$$ae \sin E . \dot{E} = \frac{e\sin\theta}{l} h$$

$$\Rightarrow a \sin E . \dot{E} = \frac{\sin\theta}{l} \sqrt{\mu l}$$

$$= \frac{\sin\theta}{b^2/a} \sqrt{\mu \frac{b^2}{a}}$$

$\Big| \; l = \frac{b^2}{a}$

$$= \sin\theta \sqrt{\frac{a\mu}{b^2}}$$

$$= \sin\theta \sqrt{\frac{a\mu}{a^2(1-e^2)}}$$

$$= \sin\nu \sqrt{\frac{\mu}{a(1-e^2)}}$$

$$\Rightarrow \dot{E} = \frac{\sqrt{\mu}}{\sqrt{a(1-e^2)}} \sin\nu \, \frac{1}{a\sin E}$$

$$= \frac{\sqrt{\mu}}{\sqrt{a}} \sin\nu \, \frac{1}{a\sqrt{1-e^2}\,\sin E}$$

$$= \sqrt{\frac{\mu}{a}} \, \frac{\sin\nu}{r\sin\nu} \quad [\text{using } ②]$$

$$= \sqrt{\frac{\mu}{a}} \cdot \frac{1}{r}$$

$$= \sqrt{\frac{\mu}{a}} \cdot \frac{1}{a(1-e\cos E)} \quad [\text{using } ③]$$

$$\Rightarrow (1-e\cos E)\, dE = \sqrt{\frac{\mu}{a^3}}\, dt$$

Integrating, $E - e\sin E = \sqrt{\frac{\mu}{a^3}}\, t + c$

But at the perihelion, $E=0$, $t=0$

$$c = 0$$

$$E - e\sin E = \sqrt{\frac{\mu}{a^3}}\, t \rightarrow ⑧$$

Now ⑤ $\Rightarrow$ $\sqrt{\mu} = \frac{2\pi}{T} a^{3/2}$

$$= n a^{3/2}$$

$$\Rightarrow \sqrt{\frac{\mu}{a^3}} = n \rightarrow (9)$$

~~ts~~ using (9), (8) becomes

$$E - e\sin E = nt = m$$

$$\boxed{E - e\sin E = m}$$

This eqn is known as the Kepler's Equation.