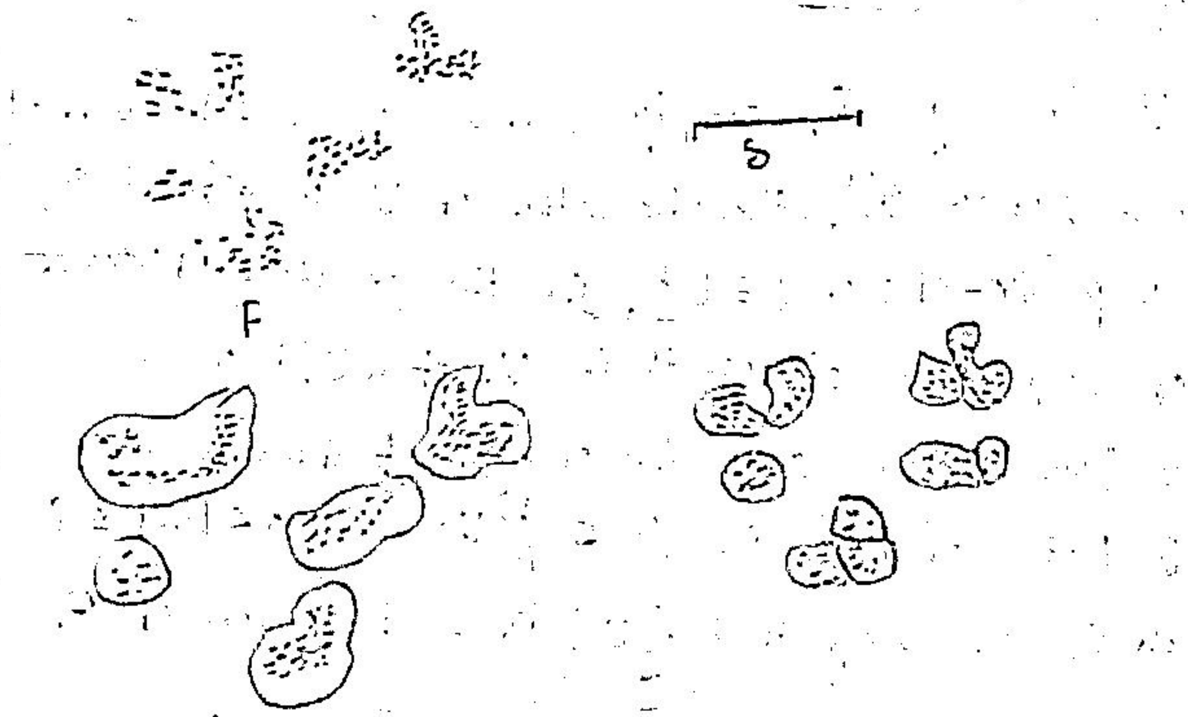
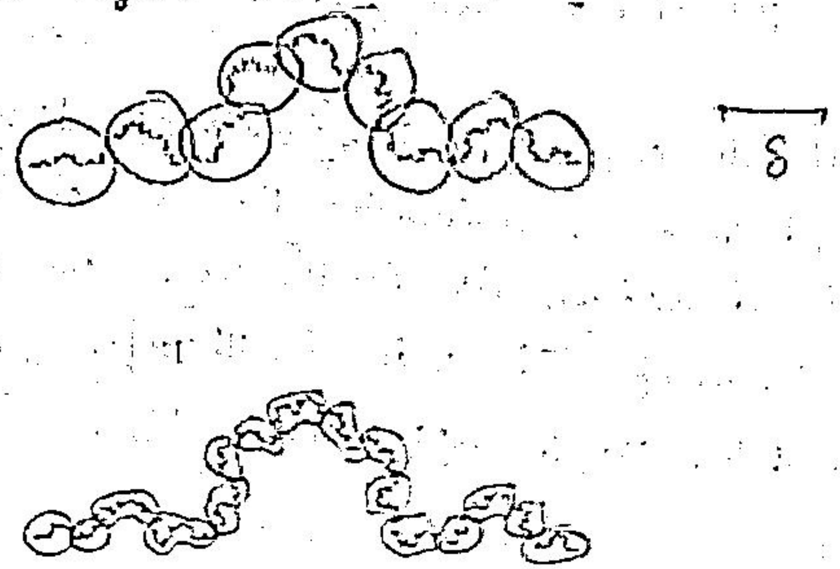
A set F and two possible δ-covers for F are given below.



The infimum of $\sum |U_i|^s$ over all such δ-cover $\{U_i\}$ gives $H^s_\delta(F)$.

Ex:



von Koch curve and two possible δ-covers.

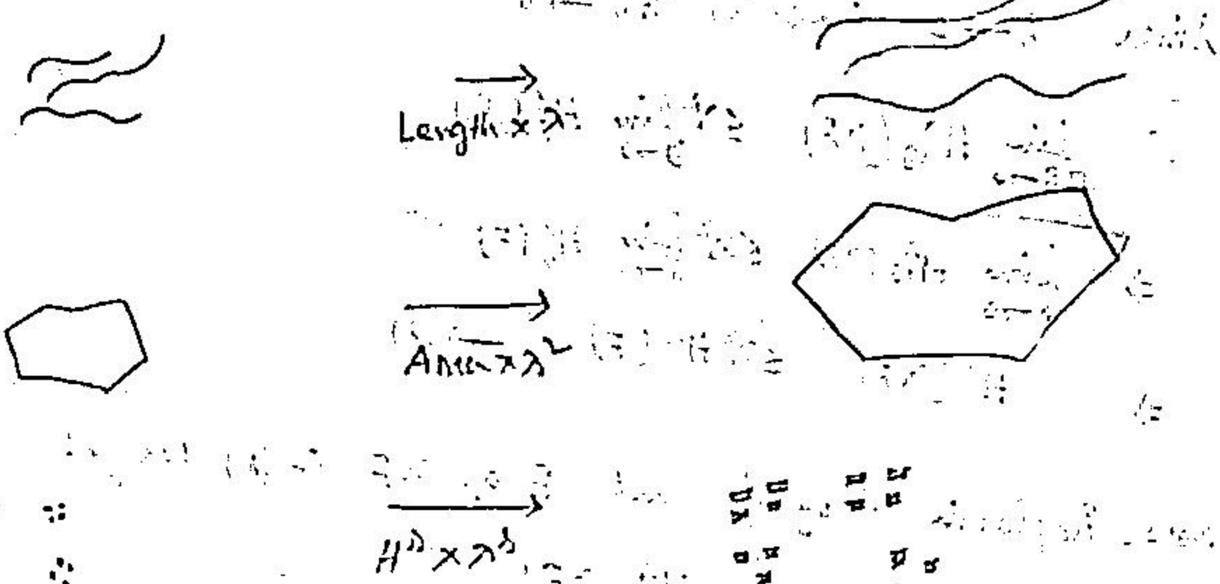
Now we can easily shown that H^s is a measure since

(i) $H^s(\phi) = 0$

(ii) if E contained in F then $H^s(E) \le H^s(F)$

(iii) $H^s\left(\bigcup_{i=1}^{\infty} F_i\right) \le \sum_{i=1}^{\infty} H^s(F_i)$, where F_i are countable on finite sequence of sets.

The scaling properties of length, area and volume are well known. On magnification by a factor λ, the length of a curve is multiplied by λ, then area of a plane region is multiplied by λ^2 and the volume of a 3-dimensional object is multiplied by λ^3. Likewise, s-dimensional Hausdorff measure scales with a factor λ^s.



Scaling sets by a factor λ.

Proposition: If $F \subset \mathbb{R}^n$ and $\lambda > 0$ then.

$$H^s(\lambda F) = \lambda^s H^s(F)$$

where $\lambda F = \{\lambda x : x \in F\}$ i.e the set scaled by a factor λ.

proof: Let $\{U_i\}$ is a δ-cover of F.

then $\{\lambda U_i\}$ is a $\lambda\delta$ - cover of λF

$$\therefore \quad H^s_{\lambda\delta}(\lambda F) \le \sum |\lambda U_i|^s$$

$$\Rightarrow H^s_{\lambda\delta}(\lambda F) \le \lambda^s \sum |U_i|^s$$

$$\Rightarrow H^s_{\lambda\delta}(\lambda F) \le \lambda^s \inf \{\sum |U_i|^s\}$$

$$\Rightarrow H^s_{\lambda\delta}(\lambda F) \le \lambda^s H^s_\delta(F)$$

Since $\delta \to 0$ implies $\lambda\delta \to 0$

$$\therefore \lim_{\lambda\delta \to 0} H^s_{\lambda\delta}(\lambda F) \le \lambda^s \lim_{\delta \to 0} H^s_\delta(F)$$

$$\Rightarrow \lim_{\lambda\delta \to 0} H^s_{\lambda\delta}(\lambda F) \le \lambda^s \lim_{\delta \to 0} H^s_\delta(F)$$

$$\Rightarrow H^s(\lambda F) \le \lambda^s H^s(F) \longrightarrow (I)$$

Now, Replacing λ by $\frac{1}{\lambda}$ and F by λF in (I) we get.

$$H^s\left(\frac{1}{\lambda} \cdot \lambda F\right) \le \frac{1}{\lambda^s} \cdot H^s(\lambda F)$$

$$\Rightarrow H^s(F) \le \frac{1}{\lambda^s} H^s(\lambda F)$$

$$\Rightarrow H^s(\lambda F) \ge \lambda^s H^s(F) \longrightarrow (II)$$

$\therefore$ (I) and (II) $\Rightarrow$ $H^s(\lambda F) = \lambda^s H^s(F)$.