Energy Integral Equation

In steady state the equation of motion for the flow in the boundary layer are

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = U\frac{\partial U(x)}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \longrightarrow (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (2)$$

Subject to the boundary condition

$$\left.\begin{array}{ll} u = v = 0 & \text{at } y = 0 \\ u = U(x) & \text{at } y = \delta \end{array}\right\} \longrightarrow (3)$$

In 1948, K. Weighardt deduced energy integral equation for laminar boundary layer. This equation is obtained by multiplying equation (1) by u on both sides and then integrating w.r.to y accross the boundary layer.

$$\int_0^\delta u^2\frac{\partial u}{\partial x}dy + \int_0^\delta u v\frac{\partial v}{\partial y}dy = \int_0^\delta u U\frac{dU}{dx}dy + \nu\int_0^\delta u\frac{\partial^2 u}{\partial y^2}dy$$

From equation (2) we have,

$$v = -\int_0^y \frac{\partial u}{\partial x}dy$$

We have, $$\int_0^\delta u^2\frac{\partial u}{\partial x}dy - \int_0^\delta u\frac{\partial u}{\partial y}\left\{\int_0^y \frac{\partial u}{\partial x}dy\right\}dy = \int_0^\delta u v\frac{dU}{dx}dy + \nu\int_0^\delta u\frac{\partial^2 u}{\partial y^2}dy.$$

$$\Rightarrow \int_0^\delta \left[u^2\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y}\left\{\int_0^y \frac{\partial u}{\partial x}dy\right\} - u v\frac{dv}{dx}\right]dy = \nu\int_0^\delta u\frac{\partial^2 u}{\partial y^2}dy \rightarrow (4)$$

Now, 2^{nd} term in the L.S. of equation (4) can be written as

$$\int_0^\delta u\frac{\partial u}{\partial y}\left\{\int_0^y \frac{\partial u}{\partial x}dy\right\}dy = \int_0^\delta \frac{1}{2}\frac{\partial u^2}{\partial y}\left\{\int_0^y \frac{\partial u}{\partial x}dy\right\}dy \quad ; \quad u\frac{du}{dy} = \frac{1}{2}\frac{\partial u^2}{\partial y}$$

$$= \left[\frac{1}{2}u^2\int_0^y \frac{\partial u}{\partial x}dy\right]_0^\delta - \int_0^\delta \frac{d}{dy}\left\{\int_0^y \frac{\partial u}{\partial x}dy\right\}\frac{1}{2}u^2 dy$$

$$= \frac{1}{2}\int_0^\delta U^2\frac{\partial u}{\partial x}dy - \frac{1}{2}\int_0^\delta u^2\frac{\partial u}{\partial x}dy$$

$$= \frac{1}{2}\int_0^\delta (U^2-u^2)\frac{\partial u}{\partial x}dy$$

whereas by combining 1st and 3rd terms in the L.S. of (IV), We get,

$$\int_0^\delta u^2\frac{\partial u}{\partial x}dy - \int_0^\delta uU\frac{dU}{dx}dy$$

$$= \int_0^\delta u\left[u\frac{\partial u}{\partial x} - U\frac{dU}{dx}\right]dy$$

$$= \frac{1}{2}\int_0^\delta u\frac{\partial}{\partial x}(u^2-U^2)dy$$

$$\text{L.S. of (IV)} = -\frac{1}{2}\int_0^\delta (U^2-u^2)\frac{\partial u}{\partial x}dy - \frac{1}{2}\int_0^\delta u\frac{\partial}{\partial x}(U^2-u^2)dy$$

$$= -\frac{1}{2}\int_0^\delta \frac{\partial}{\partial x}\left[u(U^2-u^2)\right]dy$$

$$= -\frac{1}{2}\frac{d}{dx}\int_0^\delta u(U^2-u^2)dy$$

Now, from R.S. of (IV), we have

$$\int_0^\delta u\frac{\partial^2 u}{\partial y^2}dy = \int_0^\delta \underset{\mathrm{I}}{u}\underbrace{\left\{\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\right\}}_{\mathrm{II}}dy$$

$$= \left[u \frac{\partial u}{\partial y}\right]_0^\delta - \int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy$$

$$= -\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy$$

Equation (IV) becomes,

$$-\frac{1}{2}\frac{d}{dx}\int_0^\delta u(U^2-u^2)\,dy = -\nu\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy$$

$$\text{i.e. } \frac{1}{2}\frac{d}{dx}\int_0^\delta u(U^2-u^2)\,dy = \nu\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy \longrightarrow (V)$$

This is required energy integral equation

The upper limit of integration could be replaced by $y \to \infty$, since the integrand becomes equal to zero outside the boundary layer. Thus we have,

$$\frac{1}{2}\frac{d}{dx}\int_0^\infty u(U^2-u^2)\,dy = \nu\int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy \longrightarrow (VI)$$

The quantity $\mu\left(\frac{\partial u}{\partial y}\right)^2$ represents the energy per unit volume and time which is transformed into heat by friction. The term $\frac{\rho}{2}(U^2-u^2)$ on the L.H.S represents the loss ~~of~~ in mechanical energy taking place in the boundary layer as compared to the potential flow. Hence the term $\frac{\rho}{2}\int_0^\infty u(U^2-u^2)\,dy$ represents the flux of dissipated energy and the L.H.S. represents the ~~ratio~~ rate of change of the flux. of dissipated energy per unit length in the x-direction.

Now, we define the energy thickness δ_3 (or dissipation energy) as

$$U^3\delta_3 = \int_0^\infty u(U^2-u^2)\,dy$$

or, $$\delta_3 = \int_0^\infty \frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy$$

Thus equation (VI) becomes,

$$\frac{1}{2}\frac{d}{dx}\left(U^3\delta_3\right) = \nu\int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy$$

or, $$\frac{d}{dx}\left(U^3\delta_3\right) = 2\nu\int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy$$

$$= \frac{2D}{\rho}, \quad \text{where } D = \mu\int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy$$

This is energy integral equation for 2D laminer boundary layer equation.

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§: Pohlhauson's method:

One of the earlist, until recently, most widely and used approximate methods for the solution of the boundary layer equation is that developed Pohlhausen. This method is based on the momentum integral equation of Kármán (1921) which is obtained by integrating the boundary layer equation w.r.to y accross the boundary layer.

The steady momentum integral equations of the present problem is

$$\frac{d}{dx}\int_0^\delta u(U-u)\,dy = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} \longrightarrow (1)$$

where δ is the width of the boundary Layer and $\delta = \delta(x)$.

The essence of Pohlhausin type approximate method consists in assuming a suitable expression of the velocity distribution u in the boundary Layer, taking care that it satisfies the equation (1) and then solve the resulting equation for $\delta(x)$.

For the sake of simplicity, we write,

$$u = U F(\eta) \longrightarrow (2), \text{ where } \eta = \frac{y}{\delta}.$$

Here F is some suitable non-dimensional function which can make u to satisfy the boundary conditions of (1)

i.e. $$\left.\begin{array}{ll} u=0,\ \frac{\partial u}{\partial y} \neq 0 & \text{at } y=0 \\ u=U,\ \frac{\partial u}{\partial y} = 0 & \text{at } y=\delta \end{array}\right\} \longrightarrow (3)$$

$$\int_{y=0}^{\delta} u(U-u)\,dy = \int_{\eta=0}^{1} U F(\eta)\left[U - UF(\eta)\right]\delta\, d\eta$$

$$= \int_{\eta=0}^{1} U^2 \delta \left[F(1-F)\right] d\eta$$

; $\eta = \frac{y}{\delta}$, $\delta\eta = \frac{1}{\delta} dy$, $\delta\, d\eta = dy$, $\frac{du}{dy} = \frac{\partial}{\partial\eta}(UF)\frac{\partial\eta}{dy} = UF' \cdot \frac{1}{\delta}$

$$\Rightarrow \int_{y=0}^{\delta} u(U-u)\,dy = U^2 \delta \alpha_1, \text{ where } \alpha_1 = \int_{\eta=0}^{1} F(1-F)\,d\eta \longrightarrow (4)$$

Again, $$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \left[UF'(\eta)\frac{1}{\delta}\right]_{\eta=0}$$

$$= UF'(0)\frac{1}{\delta}$$

$$= \beta_1 \frac{U}{\delta}, \text{ where } \beta_1 = F'(0) \longrightarrow (5)$$

Hence, the equation (1) becomes,

$$\frac{d}{dx}\left[U^2 \delta \alpha_1\right] = \nu \beta_1 \frac{U}{\delta}$$

$$\Rightarrow U^2 \alpha_1 \frac{d\delta}{dx} = \nu \beta_1 \frac{U}{\delta}$$

$$\Rightarrow \delta \frac{d\delta}{dx} = \nu \beta_1 \frac{1}{U\alpha_1}$$

$$\Rightarrow \frac{1}{2} \frac{d\delta^2}{dx} = \nu \beta_1 \frac{1}{U\alpha_1}$$

$$\Rightarrow \frac{d\delta^2}{dx} = 2 \frac{\nu}{\beta_1} \frac{1}{U\alpha_1}$$

$$\Rightarrow d\delta^2 = 2\nu\beta_1 \frac{1}{U\alpha_1} dx$$

Integrating, we get,

$$\delta^2 = 2\nu\beta_1 \frac{1}{U\alpha_1} x + C$$

But at $x = 0$, $\delta = 0$ hence we have $C = 0$

$$\therefore \delta^2 = \frac{2\nu\beta_1}{U\alpha_1} x$$

i.e. $$\delta = \sqrt{\frac{2\beta_1}{\alpha_1}} \sqrt{\frac{\nu x}{U}} \longrightarrow (6)$$

The shearing stress at distance x from the leading edge is given by

$$\sigma_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} \quad ; \qquad \nu = \frac{\mu}{\rho} \Rightarrow \rho\nu = \mu$$

$$= \rho\nu \frac{U\beta_1}{\delta}$$

$$= \rho\nu U\beta_1 \sqrt{\frac{\alpha_1}{2\beta_1}} \cdot \sqrt{\frac{U}{\nu x}}$$

$$= \rho\nu U \sqrt{\frac{\alpha_1\beta_1}{2}} \sqrt{\frac{U}{\nu x}}.$$

Hence the drag exerted on the two sides of a plate of length 'l' and width 'b' is given by

$$D = 2b \int_{x=0}^{l} \sigma_0 \, dx$$

$$= 2b \int_0^l \rho \nu U \sqrt{\frac{\alpha_1 \beta_1}{2}} \sqrt{\frac{U}{\nu x}} \, dx$$

$$= 2b\rho\nu U \sqrt{\frac{\alpha_1\beta_1}{2}} \sqrt{\frac{U}{\nu}} \, \frac{l^{1/2}}{\frac{1}{2}}$$

$$= 2b\rho\nu U \sqrt{2\alpha_1\beta_1} \sqrt{\frac{U}{\nu}\, l}$$

$$= 2b\rho \sqrt{2\alpha_1\beta_1} \sqrt{U^3 \nu l}$$

For the sake of comparison with the exact solution, we need to determine the displacement thickness δ_1 defined by

$$U\delta_1 = \int_{y=0}^{\delta} (U-u)\, dy \;; \quad \eta = \frac{y}{\delta} \;; \quad \left| \begin{array}{l} \delta\eta = \frac{1}{\delta} dy \\ \Rightarrow \delta d\eta = dy \end{array} \right.$$

$$= \delta \int_{\eta=0}^{1} \{U - UF(\eta)\} d\eta$$

$$= U\delta \int_{\eta=0}^{1} (1-F)\, d\eta$$

$$\Rightarrow \delta_1 = \delta \int_{\eta=0}^{1} (1-F)\, d\eta$$

$$= \alpha_2 \delta \, , \text{ where } \alpha_2 = \int_{\eta=0}^{1} (1-F)\, d\eta$$

$$\Rightarrow \delta_1 = 1.78 \sqrt{\frac{\nu x}{U}}$$

In the case of exact solution we have this value as

$$\delta_E = 1.7208 \sqrt{\frac{\nu x}{U}}$$

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