Ex. Find the displacement thickness δ_1, momentum thickness δ_2 for the velocity distribution in the boundary layer given by

$$\frac{u}{U} = \frac{y}{\delta}$$

with usual meaning of the symbols. Also calculate the value of $\frac{\delta_1}{\delta_2}$

Sol^n: Given $\frac{u}{U} = \frac{y}{\delta}$

The displacement thickness is given by

$$\delta_1 = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$

$$= \int_0^{\delta} \left(1 - \frac{y}{\delta}\right) dy$$

$$= \left[y - \frac{y^2}{2\delta}\right]_0^{\delta} = \delta - \frac{\delta^2}{2\delta} = \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

The momentum thickness δ_2 is defined by

$$\delta_2 = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$= \int_0^{\delta} \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy$$

$$= \int_0^{\delta} \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy$$

$$= \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^{\delta} = \frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2} = \frac{\delta}{2} - \frac{\delta}{3} = \frac{\delta}{6}.$$

$$\therefore \frac{\delta_1}{\delta_2} = \frac{\delta}{2} \times \frac{6}{\delta} = 3.$$

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Ex. Find the displacement thickness and the momentum thickness for the velocity distribution in the boundary layer given by

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$

Also find the ratio of δ_1 and δ_2.

Sol^n Hints $\delta_1 = \frac{\delta}{3}$, $\delta_2 = \frac{2\delta}{15}$

$$\therefore \frac{\delta_1}{\delta_2} = 2.5.$$

Ex. Show that

(i) $\int_0^\delta \frac{u}{U}\,dy = \delta - \delta_1$

(ii) $\int_0^\delta \left(\frac{u}{U}\right)^2 dy = \delta - \delta_1 - \delta_2$

(iii) $\int_0^\delta \left(\frac{u}{U}\right)^3 dy = \delta - \delta_1 - \delta_3$.

with usual meaning of the symbols.

Sol^n.

(i) By definition of the displacement thickness δ_1, we have

$$\delta_1 = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

$$= \int_0^\delta 1 \cdot dy - \int_0^\delta \frac{u}{U}\,dy$$

$$= \delta - \int_0^\delta \frac{u}{U}\,dy$$

$$\Rightarrow \int_0^\delta \frac{u}{U}\,dy = \delta - \delta_1$$

(ii) By definition of momentum thickness δ_2, we have,

$$\delta_2 = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

$$= \int_0^\delta \frac{u}{U}dy - \int_0^\delta \left(\frac{u}{U}\right)^2 dy$$

$$= \delta-\delta_1 - \int_0^\delta \left(\frac{u}{U}\right)^2 dy$$

$$\Rightarrow \int_0^\delta \left(\frac{u}{U}\right)^2 dy = \delta-\delta_1-\delta_2 .$$

(iii) By definition of energy thickness δ_3, we have,

$$\delta_3 = \int_0^\delta \frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy$$

$$= \int_0^\delta \frac{u}{U}dy - \int_0^\delta \left(\frac{u}{U}\right)^3 dy$$

$$= \delta-\delta_1 - \int_0^\delta \left(\frac{u}{U}\right)^3 dy$$

$$\Rightarrow \int_0^\delta \left(\frac{u}{U}\right)^3 dy = \delta-\delta_1-\delta_3 .$$

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Ex. Show that for a flate plate of length l, placed length wis in a uniform stream, the assumption

$$u = Uy\,(2\delta-y)/\delta^2$$

in Kármáns integral condition leads to

$$\delta = \left(\frac{30\nu x}{U}\right)^{1/2}$$

with a frictional resistance $8\sqrt{(\mu\rho l U^3)/30}$

where U is the mean stream velocity.

Sol^n. Kármán's momentum integral equation with uniform stream is

$$\frac{d}{dx}\int_0^\delta u(U-u)\,dy = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= \frac{\sigma_0}{\rho} \quad \text{where } \sigma_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \text{ is the shearing stress.}$$

i.e. $$\frac{\sigma_0}{\rho} = u\frac{d}{dx}\int_0^\delta U\,dy - \frac{d}{dx}\int_0^\delta u^2\,dy$$

$$= U^2\frac{d}{dx}\int_0^\delta y(2\delta-y)/\delta^2\,dy - U^2\frac{d}{dx}\int_0^\delta y^2(2\delta-y)^2/\delta^4\,dy$$

$$= U^2\frac{d}{dx}\left[\frac{1}{\delta^2}\left(2\delta\frac{y^2}{2}-\frac{y^3}{3}\right)\right]_0^\delta - U^2\frac{d}{dx}\int_0^\delta y^2(4\delta^2-4\delta y+y^4)/\delta^4\,dy.$$

$$\Rightarrow \frac{\sigma_0}{\rho} = U^2\frac{d}{dx}\left[\delta-\frac{\delta}{3}\right] - U^2\frac{d}{dx}\left[\left(4\delta^2\frac{y^3}{3}-4\delta\frac{y^4}{4}+\frac{y^5}{5}\right)/\delta^4\right]_0^\delta$$

$$= U^2\frac{d}{dx}\left(\frac{2}{3}\delta\right) - U^2\frac{d}{dx}\left[\delta\left(\frac{4}{3}-1+\frac{1}{5}\right)\right]$$

$$= \frac{2}{3}U^2\frac{d\delta}{dx} - \frac{8}{15}U^2\frac{d\delta}{dx}$$

$$= \frac{2}{15}U^2\frac{d\delta}{dx}$$

$$\Rightarrow \sigma_0 = \frac{2}{15}\rho U^2\frac{d\delta}{dx} \longrightarrow (1)$$

By definition, $\sigma_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$

$$= \mu\left[\frac{\partial}{\partial y}\left\{U y(2\delta-y)/\delta^2\right\}\right]_{y=0}$$

$$= \frac{\mu U}{\delta^2}\left[2\delta-y\right]_{y=0}$$

$$\Rightarrow \sigma_0 = \frac{2\mu U}{\delta} \longrightarrow (2)$$

∴ From equation (1) and (2) we have

$$\frac{2}{15}\rho U^2 \frac{d\delta}{dx} = \frac{2\mu U}{\delta}$$

$$\Rightarrow 2\delta \frac{d\delta}{dx} = \frac{30\mu U}{\rho U^2}$$

$$\Rightarrow \frac{d\delta^2}{dx} = \frac{30\nu}{U} \quad , \quad \nu = \frac{\mu}{\rho}$$

$$\Rightarrow d\sigma^2 = \frac{30\nu}{U} dx$$

Integrating, we get,

$$\delta^2 = \frac{30\nu}{U} x + c \longrightarrow (3)$$

But at $x = 0$, $\delta = 0$ therefore from (3), we have,

$$c = 0$$

$$\therefore \quad \delta^2 = \frac{30\nu}{U} x$$

$$\Rightarrow \delta = \sqrt{\frac{30\nu x}{U}}$$

Now, total frictional resistance for a plate of length l and unit breadth is

$$D = 2\int_{x=0}^{l} \sigma_0 \, dx$$

$$= 2\int_{x=0}^{l} 2\mu \frac{U}{\delta} dx$$

$$= 4\mu U \int_0^l \frac{dx}{\delta}$$

$$= 4\mu U \int_0^l \left(\frac{30\nu x}{U}\right)^{-\frac{1}{2}} dx$$

$$= 4\mu U \int_0^l \left(\frac{U}{30\nu x}\right)^{\frac{1}{2}} dx$$

$$= \frac{40\mu U^{3/2}}{(30\nu)^{\frac{1}{2}}} \cdot 2\left[x^{\frac{1}{2}}\right]_0^l$$

$$= \frac{8\mu U^{3/2}}{30\nu} \cdot \frac{l^{1/2}}{\frac{1}{2}}$$

$$= 8\cdot\left(\frac{\rho U^3 \mu l}{30}\right)^{1/2}.$$

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Ex. A ~~platae~~ plate of length l is placed in uniform stream in the direction of its length. Discuss the possibility of the assumption

$$u = U\sin\left(\frac{\pi y}{2\delta}\right)$$

for the velocity in the boundary layer. Calculate the thickness of the boundary layer with the assumption and determine the frictional resistance on the plate.

Hints:

$$\delta^2 = \frac{2\pi^2 \nu x}{(4-\pi)U} = 4.8\sqrt{\frac{\nu x}{U}}$$

$$\tau_0 = \mu U\left[\frac{(4-\pi)U}{2\nu}\right]^{\frac{1}{2}} \cdot 2l^{1/2}$$

$$= \left[2(\gamma-\mu)U^3 l\rho\mu\right]^{1/2}.$$

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