unit (2)

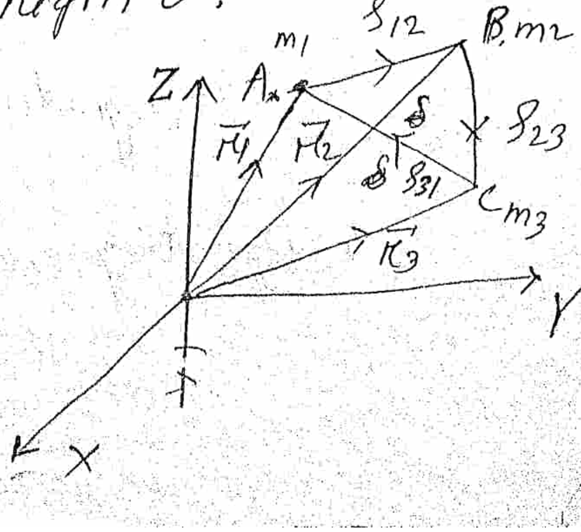
~~Theree~~ body problem :-

The general problem of the motion of three bodies (assumed to be point masses) subject only to their mutual ~~gravity~~ gravitational attraction is called three body problem.

Three bodies each being attracted by their mutual gravitational attraction, to determine the motion of the bodies which move freely in space in any manner initially

2017 (10)

Let m_1, m_2, m_3 be the masses of the bodies with p.v. s $\vec{r_1}, \vec{r_2}, \vec{r_3}$ respectively w.r.to some fixed origin O.



Let $\vec{AB} = \vec{\varrho_{12}} = \vec{r_2} - \vec{r_1}$

$\vec{BC} = \vec{\varrho_{23}} = \vec{r_3} - \vec{r_2}$

$\vec{CA} = \vec{\varrho_{31}} = \vec{r_1} - \vec{r_3}$

The eqn's of motion of the mass m_1 by Newton's law is

$$m_1\ddot{\vec{r}}_1 = K^2\left[\frac{m_1m_2}{\rho_{12}^2}\frac{\vec{\rho}_{12}}{\rho_{12}} + \frac{m_1m_3}{\rho_{13}^2}\frac{\vec{\rho}_{13}}{\rho_{13}}\right]$$

$$= K^2\left[\frac{m_1m_2}{\rho_{12}^3}\vec{\rho}_{12} - \frac{m_1m_3}{\rho_{13}^3}\vec{\rho}_{31}\right] \rightarrow (1)$$

Similarly, $m_2\ddot{\vec{r}}_2 = K^2\left[\frac{m_2m_3}{\rho_{23}^3}\vec{\rho}_{23} - \frac{m_2m_1}{\rho_{21}^3}\vec{\rho}_{12}\right] \rightarrow (2)$

$$m_3\ddot{\vec{r}}_3 = K^2\left[\frac{m_3m_1}{\rho_{31}^3}\vec{\rho}_{31} - \frac{m_3m_2}{\rho_{23}^3}\vec{\rho}_{23}\right] \rightarrow (3)$$

Now adding (1) (2) and (3) we get

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 + m_3\ddot{\vec{r}}_3 = 0$$

Integrating $m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 + m_3\dot{\vec{r}}_3 = c_1$

Integrating, we get

$$m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 = \vec{c}_1 t + \vec{c}_2 \rightarrow (4)$$

where $\vec{c_1}$ and $\vec{c_2}$ are constant vectors of integration.

If $\vec{R}$ be the position vector of the centre of mass of the particles, then

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3}}{m_1 + m_2 + m_3}$$

$$\therefore (4) \Rightarrow \vec{R} = \frac{\vec{c_1}t + \vec{c_2}}{M}, \text{ where } M = m_1 + m_2 + m_3$$

$$\Rightarrow R = \frac{\vec{c_1}.t}{M} + \frac{\vec{c_2}}{M} \longrightarrow (5)$$

which is called the first integral of the equation of motion consisting of six arbitrary constant given by the components of $\vec{c_1}$ and $\vec{c_2}$

Eqn (5) shows that the centre of mass of the three bodies is at rest or moves with constant velocity along a straight line.

Again taking the cross product of (1), (2) and (3) with $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$ respectively, we get

$$\vec{r_1} \times m_1\ddot{\vec{r_1}} = k^2\left[\frac{m_1 m_2}{\rho_{12}^3}\vec{r_1}\times\vec{\rho_{12}} - \frac{m_1 m_3}{\rho_{13}^3}\vec{r_1}\times\vec{\rho_{31}}\right] \longrightarrow (6)$$

$$\vec{r_2} \times m_2\ddot{\vec{r_2}} = k^2\left[\frac{m_2m_3}{\rho_{23}^3}\vec{r_2}\times\vec{\rho_{23}} - \frac{m_1m_2}{\rho_{12}^3}\vec{r_2}\times\vec{\rho_{12}}\right] \rightarrow (7)$$

$$\vec{r_3} \times m_3\ddot{\vec{r_3}} = k^2\left[\frac{m_3m_1}{\rho_{31}^3}\vec{r_3}\times\vec{\rho_{31}} - \frac{m_2m_3}{\rho_{23}^3}\vec{r_3}\times\vec{\rho_{23}}\right] \rightarrow (8)$$

adding (6), (7) and (8), we get

$$\sum_{i=1}^{3}\vec{r_i}\times m_i\ddot{\vec{r_i}} = k^2\left[\frac{m_1m_2}{\rho_{12}^3}(\vec{r_1}-\vec{r_2})\times\vec{\rho_{12}} + \frac{m_2m_3}{\rho_{23}^3}(\vec{r_2}-\vec{r_3})\times\vec{\rho_{23}} + \frac{m_3m_1}{\rho_{31}^3}(\vec{r_3}-\vec{r_1})\times\vec{\rho_{31}}\right]$$

$$= k^2\left[-\frac{m_1m_2}{\rho_{12}^3}\vec{\rho_{12}}\times\vec{\rho_{12}} - \frac{m_2m_3}{\rho_{23}^3}\vec{\rho_{23}}\times\vec{\rho_{23}} - \frac{m_3m_1}{\rho_{31}^3}\vec{\rho_{31}}\times\vec{\rho_{31}}\right]$$

$$= 0$$

$$\Rightarrow \frac{d}{dt}\sum_{i=1}^{3}\vec{r_i}\times m_i\dot{\vec{r_i}} = 0 \qquad \left[\because \dot{\vec{r_i}}\times\dot{\vec{r_i}} = 0\right.$$

$$\Rightarrow \sum_{i=1}^{3}\vec{r_i}\times m_i\vec{v_i} = \vec{L}$$, a constant vector $\rightarrow (9)$

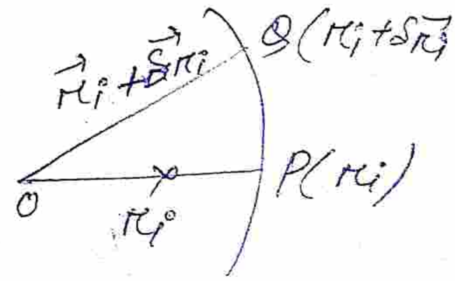
(Integrating)

This shows that the total angular momentum of the system consisting of three bodies about O is conserved. This is called the 2nd integral of the equation of motion of the three ~~bo~~ masses consisting of three arbitrary scalar constant.

The eqn (9) can be written as

$$\sum_{i=1}^{3} m_i \left(\vec{r_i} \times \vec{v_i} \right) = \overline{L} \longrightarrow (10)$$

Let P be the position of the i^{th} body at time t and $\overrightarrow{OP} = \vec{r_i}$.



Let Q be the position of the body at time ~~t+δt after~~ $t+\delta t$ after a lapse of time δt, let $\overrightarrow{OQ} = \vec{r_i} + \delta\vec{r_i}$

Let $\delta\vec{A_i}$ be the vector area swept out by the radius vector joining the ~~bo~~ i^{th} body and the point o.

$$\delta \vec{A_i} = \frac{1}{2} \vec{OP} \times \vec{OQ}$$

$$= \frac{1}{2} \vec{r_i} \times (\vec{r_i} + \delta \vec{r_i})$$

$$= \frac{1}{2} \vec{r_i} \times \delta \vec{r_i}$$

$$\therefore \frac{\delta \vec{A_i}}{\delta t} = \frac{1}{2} \vec{r_i} \times \frac{\delta \vec{r_i}}{\delta t}$$

$$\Rightarrow \lim_{\delta t \to 0} \frac{\delta \vec{A_i}}{\delta t} = \frac{1}{2} \vec{r_i} \times \vec{v_i}$$

$$\Rightarrow \frac{d \vec{A_i}}{dt} = \frac{1}{2} \vec{r_i} \times \vec{v_i}$$

$$\Rightarrow \vec{r_i} \times \vec{v_i} = 2 \dot{\vec{A_i}}$$

$\therefore$ (10) reduces to $\sum_{i=1}^{3} m_i \, 2\dot{\vec{A_i}} = \vec{l}$

$$\Rightarrow \sum_{i=1}^{3} m_i \dot{\vec{A_i}} = \frac{\vec{L}}{2} = \text{a const vector} \longrightarrow (11)$$

This shows that the areal momentum of the system of masses is constant. This is called the integral of areal momentum.

Again taking dot product of ① ,② & ③ with $\dot{\vec{r_1}}, \dot{\vec{r_2}}, \dot{\vec{r_3}}$ respectively and then adding, we get

$$\sum_{i=1}^{3} m_i \dot{\vec{r_i}} \ddot{\vec{r_i}} = K^2 \left[\frac{m_1 m_2}{\rho_{12}^3} (\dot{\vec{r_1}} - \dot{\vec{r_2}}) \cdot \vec{\rho_{12}} + \frac{m_2 m_3}{\rho_{23}^3} (\dot{\vec{r_2}} - \dot{\vec{r_3}}) \vec{\rho_{23}} + \frac{m_3 m_1}{\rho_{31}^3} (\dot{\vec{r_3}} - \dot{\vec{r_1}}) \cdot \vec{\rho_{31}} \right]$$

$$= -K^2 \left[\frac{m_1 m_2}{\rho_{12}^3} \dot{\vec{\rho_{12}}} \cdot \vec{\rho_{12}} + \frac{m_2 m_3}{\rho_{23}^3} \dot{\vec{\rho_{23}}} \cdot \vec{\rho_{23}} + \frac{m_3 m_1}{\rho_{31}^3} \dot{\vec{\rho_{31}}} \cdot \vec{\rho_{31}} \right]$$

$$= -K^2 \left[\frac{m_1 m_2}{\rho_{12}^3} \dot{\rho}_{12} \rho_{12} + \frac{m_2 m_3}{\rho_{23}^3} \rho_{23} \dot{\rho}_{23} + \frac{m_3 m_1}{\rho_{31}^3} \dot{\rho}_{31} \rho_{31} \right]$$

$$= -K^2 \left[\frac{m_1 m_2}{\rho_{12}^2} \dot{\rho}_{12} + \frac{m_2 m_3}{\rho_{23}^2} \dot{\rho}_{23} + \frac{m_3 m_1}{\rho_{31}^2} \dot{\rho}_{31} \right] \rightarrow (12)$$

$$\left[\because \vec{u} \cdot \dot{\vec{u}} = u \dot{u} \right]$$

we have

$$\frac{d}{dt} (\dot{\vec{r_i}}^2) = \frac{d}{dt} (\dot{\vec{r_i}} \cdot \dot{\vec{r_i}})$$

$$= 2 \dot{\vec{r_i}} \cdot \ddot{\vec{r_i}}$$

$$\Rightarrow \dot{\vec{r_i}} \cdot \ddot{\vec{r_i}} = \frac{1}{2} \frac{d}{dt} (\dot{\vec{r_i}}^2) \rightarrow (13)$$

and $$\frac{d}{dt} \left(\frac{1}{u} \right) = - \frac{1}{u^2} \frac{du}{dt} = - \frac{1}{u^2} \dot{u}.$$

$$\Rightarrow \frac{\dot{u}}{u^2} = - \frac{d}{dt} \left(\frac{1}{u} \right) \rightarrow (14)$$

(12) $\Rightarrow \sum_{i=1}^{3} m_i \frac{1}{2} \frac{d}{dt}(\dot{\vec{r}_i}^2) = -k^2\Big[-m_1m_2\frac{d}{dt}\Big(\frac{1}{\rho_{12}}\Big) - m_2m_3\frac{d}{dt}\Big(\frac{1}{\rho_{23}}\Big) - m_3m_1\frac{d}{dt}\Big(\frac{1}{\rho_{31}}\Big)\Big]$

$$\Rightarrow \sum_{i=1}^{3} \frac{d}{dt}\Big(\frac{1}{2}m_i\dot{\vec{r}_i}^2\Big) = k^2\Big[\frac{d}{dt}\Big(\frac{m_1m_2}{\rho_{12}}\Big) + \frac{d}{dt}\Big(\frac{m_2m_3}{\rho_{23}}\Big) + \frac{d}{dt}\Big(\frac{m_3m_1}{\rho_{31}}\Big)\Big]$$

$$\Rightarrow \frac{d}{dt}\Big(\sum_{i=1}^{3}\frac{1}{2}m_i\dot{\vec{r}_i}^2\Big) = -\frac{d}{dt}\Big[-k^2\frac{m_1m_2}{\rho_{12}} - k^2\frac{m_2m_3}{\rho_{23}} - k^2\frac{m_3m_1}{\rho_{31}}\Big]$$

$$\Rightarrow \frac{d}{dt}(T) = -\frac{dV}{dt}$$

where $T = \sum_{i=1}^{3}\frac{1}{2}m_i\dot{\vec{r}_i}^2 = $ K.E of the system consisting of three masses

and $V = -k^2\Big(\frac{m_1m_2}{\rho_{12}} + \frac{m_2m_3}{\rho_{23}} + \frac{m_3m_1}{\rho_{31}}\Big) =$ P.E of the system.

$$\frac{d}{dt}(T+V) = 0$$

$\Rightarrow T+V = $ a constant $= E$ (say)

which is called the integral of energy.

it involves one arbitrary scalar constant E.

The eqn (15) expresses the conservation of energy for the system of three masses

Thus we have got two vector integrals (6) & (7) and energy integral (15) consisting of $(6+3+1=)$ 10 arbitrary constants but the general sol^n of (1) (2) & (3) should have $(6 \times 3 =)$ 18 arbitrary const.

Hence the eq^n deduced cannot represent the general sol^n, so the problem remains unsolvable. It is possible only to find particular sol^n under certain restriction.