Ex. If a polynomial of degree 2 (or 3) is assumed for the velocity profile in the boundary layer, determine the two constants from the appropriate boundary conditions. Also, determine the boundary layer thickness, displacement thickness, momentum thickness and shearing stress at the wall.

Soln. Let us consider the profile of velocity be of the form

$$\frac{u}{U} = a + b\eta + c\eta^2 \longrightarrow (i)$$

where $\eta = \frac{y}{\delta}$ and a, b, c are constants.

The boundary conditions for this problem are

$$\left.\begin{array}{lll} u=0,\ v=0 & & \text{at } y=0 \\ u=U,\ \frac{\partial u}{\partial y}=0 & & \text{at } y=\delta \end{array}\right\} \longrightarrow (ii)$$

Now, using the boundary condition

$u=0$ at $y=0$ in (i)

we have, $a=0$.

Also, the B.C., $u=U$ at $y=\delta$ gives

$$b+c=1 \longrightarrow (iii)$$

and the B.C., $\frac{\partial u}{\partial y}=0$ at $y=\delta$ gives

$$b+2c=0 \longrightarrow (iv)$$

$\therefore$ (iii) and (iv) give,

$$b=2,\ c=1.$$

$\frac{u}{U} = a + b\eta + c\eta^2$

$= 0 + b\cdot\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2$

$\frac{1}{U}\frac{\partial u}{\partial y} = \frac{b}{\delta} + 2c\frac{y}{\delta^2}$

at $y=\delta$

$0 = \frac{b}{\delta} + 2c\cdot\frac{\delta}{\delta^2}$

$\Rightarrow 0 = b + 2c$

Thus $\frac{u}{U} = 2\eta - \eta^2$

$$\Rightarrow u = U(2\eta - \eta^2) \longrightarrow (v)$$

Now, the displacement thickness is defined by

$$\delta_1 = \int_{y=0}^{\delta} \left(1 - \frac{u}{U}\right) dy \qquad , \eta = \frac{y}{\delta}$$

$$= \delta \int_{\eta=0}^{1} (1 - 2\eta + \eta^2)\, d\eta$$

$$= \delta \left[\eta - \eta^2 + \frac{\eta^3}{3}\right] = \frac{\delta}{3}.$$

The momentum thickness is defined by

$$\delta_2 = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$= \delta \int_0^1 (1 - 2\eta + \eta^2)(2\eta - \eta^2)\, d\eta$$

$$= \delta \int_0^1 \left[2\eta - 4\eta^2 + 2\eta^3 - \eta^2 + 2\eta^3 - \eta^4\right] d\eta$$

$$= \delta \left[\eta^2 - \frac{5\eta^3}{3} + \frac{4\eta^4}{4} - \frac{\eta^5}{5}\right]_0^1$$

$$= \delta \left[1 - \frac{5}{3} + 1 - \frac{1}{5}\right]$$

$$= \frac{2\delta}{15}.$$

The shearing stress is given by

$$\sigma_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= \frac{\mu}{\delta} 2U.$$

;

$$u = U(2\eta - \eta^2)$$

$$\frac{\partial u}{\partial y} = \left[\frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right]_{\eta=0}$$

$$= \left[\frac{U}{\delta}(2 - 2\eta)\right]_{\eta=0}$$

$$= \frac{1}{\delta} \cdot 2U.$$

Now, we have, Kármán's momentum integral equation as

$$\frac{\sigma_0}{\rho} = \frac{d}{dx}\left(U^2\delta_2\right) \quad ;$$

Steady flow

$$\frac{\partial}{\partial x}\left(U^2\delta_2\right) + \cancel{U\delta_1\frac{\partial U}{\partial x}} = \sigma_0/\rho$$

$$\downarrow$$

$$\frac{d}{dx}$$

$$\Rightarrow \frac{1}{\rho}\frac{\mu}{\delta}2U = U^2\frac{d}{dx}\left(\frac{2}{15}\delta\right)$$

$$= \frac{2U^2}{15}\frac{d\delta}{dx}$$

$$\Rightarrow \delta\frac{d\delta}{dx} = \frac{15\mu}{\rho U} = \frac{15\nu}{U}, \quad \nu = \frac{\mu}{\rho}$$

$$\Rightarrow \frac{1}{2}\frac{d\delta^2}{dx} = \frac{15\nu}{U}$$

$$\Rightarrow \frac{d\delta^2}{dx} = \frac{30\nu}{U}$$

Integrating, $\delta^2 = \frac{30\nu}{U}x + C$

But at $x = 0$, $\delta = 0$, therefore, we have $C = 0$

Hence, we have

$$\delta^2 = \frac{30\nu x}{U}$$

$$\Rightarrow \delta = \sqrt{\frac{30\nu x}{U}}.$$

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Sol^n Hints: Polynomial $\frac{u}{U} = a + b\eta + c\eta^2 + d\eta^3$

B.C. $u = 0$ at $y = 0$

Again, from B.L. equation we have,

$$\frac{\partial^2 u}{\partial y^2} = 0 \text{ at } y = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \cancel{U\frac{dU}{dx}} + \nu\frac{\partial^2 u}{\partial y^2}$$

using $u = v = 0$ at $y = 0$

we get, $0 = \nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$.

For this problem we have, (25)

$$u = 0, \quad \frac{\partial u}{\partial y} \neq 0, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad y = 0$$

$$u = U, \quad \frac{\partial u}{\partial y} = 0 \quad \text{at } y = \delta$$

$$a = c = 0, \quad b + d = 1, \quad b + 3d = 0$$

$$\therefore\ u = U\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)$$

$$\delta_1 = \frac{3}{8}\delta, \quad \delta_2 = \frac{39}{280}\delta$$

$$\therefore\ \frac{\tau_0}{\rho} = \nu \frac{U}{\delta}\frac{3}{2}, \quad \delta = \sqrt{\frac{280}{13}\frac{\nu}{U}x}$$

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