problem: prove that $H^s : P(\mathbb{R}^n) \longrightarrow [0,\infty)$ is an outer-measure on $\mathbb{R}^n$.

proof: To prove $H^s(\phi) = 0$:

Since, $\phi \subseteq \phi$

$\therefore H^s_\delta(\phi) \leq |\phi|^s$, s is a non negative number.

$\therefore H^s_\delta(\phi) = 0$

$\therefore \lim_{\delta \to 0} H^s_\delta(\phi) = 0$

$\Rightarrow H^s(\phi) = 0$

To prove if $E \subseteq F$, $H^s(E) \leq H^s(F)$:

If $E \subseteq F$ then every δ-cover of F is also an δ cover of E, i.e. the collection of all δ-cover of F is a sub-collection of δ-covers of E.

Therefore, $H^s_\delta(E) \leq H^s_\delta(F)$

$\Rightarrow \lim_{\delta \to 0} H^s_\delta(E) \leq \lim_{\delta \to 0} H^s_\delta(F)$

$\Rightarrow H^s(E) \leq H^s(F)$.

To prove $H^s\left(\bigcup_{k=1}^{\infty} F_k\right) \leq \sum_{k=1}^{\infty} H^s(F_k)$, where $\{F_k\}$ is a sequence of subsets of $\mathbb{R}^n$:

We assume that $H^s(F_k) < \infty$.

Given $\epsilon > 0$, for any $k \in \mathbb{N}$ there exist a δ-cover

$\{U_{i,k}\}$ of F_K, such that

$$\sum_i |U_{i,k}|^s < H^s_\delta(F_K) + \frac{\epsilon}{2^K}$$

Taking summation through $K = 1, 2, 3, \ldots$

$$\sum_K \sum_i |U_{i,k}|^s < \sum_K H^s_\delta(F_K) + \sum_K \frac{\epsilon}{2^K}$$

Since $\{U_{i,k}\}$ is a cover of F_K for each k

$\therefore \sum_{K \in N} \{U_{i,k}\}$ is a cover of $\bigcup_{K \in N} F_K$

$$\therefore H^s_\delta\left(\bigcup_K F_K\right) \leq \sum_K \sum_i |U_{i,k}|^s < \sum_K H^s_\delta(F_K) + \epsilon$$

Since ϵ is arbitrary

$$\Rightarrow H^s_\delta\left(\bigcup_K F_K\right) \leq \sum_{K=1}^{\infty} H^s_\delta(F_K), \text{ for any } \delta > 0$$

Taking $\delta \to 0$, we have

$$H^s\left(\bigcup_K F_K\right) \leq \sum_K H^s(F_K)$$

If $H^s(F_K) = \infty$, for some K, then the result is obvious.

Proposition: Let $F \subseteq \mathbb{R}^n$ and $f: F \to \mathbb{R}^n$ be a mapping such that,

$$|f(x) - f(y)| \leq c|x-y|^{\alpha}, \quad x, y \in F$$

for constants $c > 0$ and $\alpha > 0$. Then for each s

$$H^{s/\alpha}(f(F)) \leq c^{s/\alpha} H^s(F)$$

Pf: Let $\{U_i\}$ is a δ-cover of F. Then, since

$$|f(F \cap U_i)| \leq c|U_i|^{\alpha}$$

$$\Rightarrow |f(F \cap U_i)| \leq c\delta^{\alpha} = \varepsilon \text{ (say)}$$

$\therefore \{f(F \cap U_i)\}$ is an ε-cover of $f(F)$.

Now, $|f(F \cap U_i)| \leq c|U_i|^{\alpha}$

$$\Rightarrow |f(F \cap U_i)|^{s/\alpha} \leq c^{s/\alpha} |U_i|^{\alpha \cdot s/\alpha}$$

$$\Rightarrow |f(F \cap U_i)|^{s/\alpha} \leq c^{s/\alpha} |U_i|^{s}$$

$$\Rightarrow \sum |f(F \cap U_i)|^{s/\alpha} \leq c^{s/\alpha} \sum |U_i|^{s}$$

$$\Rightarrow \inf \sum |f(F \cap U_i)|^{s/\alpha} \leq c^{s/\alpha} \inf \sum |U_i|^{s}$$

$$\Rightarrow H_{\varepsilon}^{s/\alpha}(f(F)) \leq c^{s/\alpha} H_{\delta}^{s}(F)$$

Now if $\delta \to 0$, then $\varepsilon = c\delta^{\alpha} \to 0$

$\therefore$ taking $\delta \to 0$ we have

$$H^{1/\kappa}(f(F)) \le C^{1/\kappa} H^1(F) ,$$