Stationary soln of the three body problem.

2015

(10) The special of three body problem called a stationary soln under certain restrictions. In the year 1977, Lagrange obtained two specials soln for a three body problem.

Stationary soln means one in which the geometrical configuration of the bodies remain unchanged for all time, assuming them to be projected initially in the same plane.

The invariance of the geometrical configuration can take place in two different ways :

(i) If the motion of the masses is such that their mutual distances from each other remains unchanged throughout the motion which can occur if they move around the centre of masses as if rigidly connected in their plane i.e the resultant force on every mass point pass through the centre of mass,

(ii) If the distances of the mass must be expanded or constructed in the same ratio.

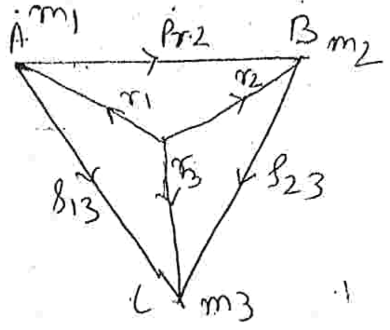
<u>case I:-</u> Let us suppose that the three masses move in a coplaner circular orbit around their centre of mass G with constant angular speed n.

Let m_1, m_2, m_3 be the masses of the bodies with p.v.s $\vec{r_1}, \vec{r_2}, \vec{r_3}$ resp. relative to G as origin.

Let $\vec{\rho}_{12} = \vec{AB} = \vec{r_2} - \vec{r_1}$

$\vec{\rho}_{23} = \vec{BC} = \vec{r_3} - \vec{r_2}$

$\vec{\rho}_{31} = \vec{CA} = \vec{r_1} - \vec{r_3}$



Now, the position vector of G relative to G = 0.

$$\Rightarrow \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3}}{m_1 + m_2 + m_3} = 0$$

$$\Rightarrow m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} = \vec{0} \rightarrow (1)$$

the eqn of motion of the masses are

$$m_1\ddot{\vec{r_1}} = k^2\left[\frac{m_1 m_2}{\rho_{12}^3}\vec{\rho}_{12} - \frac{m_1 m_3}{\rho_{31}^3}\vec{\rho}_{31}\right] \rightarrow (2)$$

$$m_2\ddot{\vec{r_2}} = k^2\left[\frac{m_2 m_3}{\rho_{23}^3}\vec{\rho}_{23} - \frac{m_2 m_1}{\rho_{12}^3}\vec{\rho}_{12}\right] \rightarrow (3)$$

$$m_3\ddot{\vec{r_3}} = k^2\left[\frac{m_3 m_1}{\rho_{13}^3}\vec{\rho}_{31} - \frac{m_3 m_2}{\rho_{23}^3}\vec{\rho}_{23}\right] \rightarrow (4)$$

Now $\ddot{\vec{r_1}}$ = acceleration of m_1 along $\vec{GA}$

$$= (\ddot{r_1} - r_1\dot{\theta}_1^2)\hat{r_1} + \frac{1}{r_1}\frac{d}{dt}(r_1^2\dot{\theta}_1)\hat{\theta}_1$$

$$= -r_1 n^2\hat{r_1} + \underbrace{\frac{1}{r_1}\frac{d}{dt}(r_1^2 n)}_{=0}\hat{\theta}_1 \quad [\because \ddot{r_1} = 0 \ \& \ \dot{\theta}_1 = n]$$

$$= -n^2 r_1\frac{\vec{r_1}}{r_1} \quad [\because n = \text{constant}, \ r_1 = \text{constant}]$$

$$= -n^2\vec{r_1}$$

similarly $\ddot{\vec{r_2}} = -n^2 \vec{r_2}$

$\ddot{\vec{r_3}} = -n^2 \vec{r_3}$

$\therefore$ (2), (3) & (4) reduces to

$$-n^2 \vec{r_1} = K^2 \left[\frac{m_2}{\rho_{12}^3} \vec{\rho_{12}} - \frac{m_3}{\rho_{31}^3} \vec{\rho_{31}} \right] \rightarrow (5)$$

$$-n^2 \vec{r_2} = K^2 \left[\frac{m_3}{\rho_{23}^3} \vec{\rho_{23}} - \frac{m_1}{\rho_{12}^3} \vec{\rho_{12}} \right] \rightarrow (6)$$

$$-n^2 \vec{r_3} = K^2 \left[\frac{m_1}{\rho_{31}^3} \vec{\rho_{31}} - \frac{m_2}{\rho_{23}^3} \vec{\rho_{23}} \right] \rightarrow (7)$$

The eq^n (5), (6), (7) satisfy the eq^n (1).

Again taking cross product of (5) with $\vec{r_1}$, we get

$$0 = K^2 \vec{r_1} \times \left[\frac{m_2}{\rho_{12}^3} (\vec{r_2} - \vec{r_1}) - \frac{m_3}{\rho_{31}^3} (\vec{r_1} - \vec{r_3}) \right]$$

$$\Rightarrow \frac{m_2}{\rho_{12}^3} \vec{r_1} \times \vec{r_2} + \frac{m_3}{\rho_{31}^3} \vec{r_1} \times \vec{r_3} = 0.$$

$$\Rightarrow \frac{\vec{r_1} \times m_2 \vec{r_2}}{\rho_{12}^3} + \frac{\vec{r_1} \times (-m_1 \vec{r_1} - m_2 \vec{r_2})}{\rho_{31}^3} = 0 \left[\because m_1 \vec{r_1} + m_2 \vec{r_2} + m_3 \vec{r_3} = 0 \right]$$

$$\Rightarrow \frac{\vec{r_1} \times m_2 \vec{r_2}}{\rho_{12}^3} - \frac{\vec{r_1} \times m_2 \vec{r_2}}{\rho_{31}^3} = 0$$

$$\Rightarrow (\vec{r_1} \times \vec{r_2})\left(\frac{1}{\rho_{12}^3} - \frac{1}{\rho_{31}^3}\right) = \vec{0} \rightarrow (8)$$

similarly, –

$$\vec{r_2} \times \vec{r_3}\left(\frac{1}{\rho_{23}^3} - \frac{1}{\rho_{12}^3}\right) = \vec{0} \rightarrow (9)$$

$$(\vec{r_3} \times \vec{r_1})\left(\frac{1}{\rho_{31}^3} - \frac{1}{\rho_{23}^3}\right) = \vec{0} \rightarrow (10)$$

from (8), we get

either $\vec{r_1} \times \vec{r_2} = 0$ or $\rho_{12} = \rho_{31}$

case (a):

Let us suppose that $\vec{r_1} \times \vec{r_2} \neq 0$.

$$\rho_{12} = \rho_{31}$$

$$(9) \Rightarrow (\vec{r_2} \times \vec{r_3})\left(\frac{1}{\rho_{23}^3} - \frac{1}{\rho_{31}^3}\right) = 0 \rightarrow (11)$$

$$(10) \Rightarrow (\vec{r_1} \times \vec{r_3})\left(\frac{1}{\rho_{23}^3} - \frac{1}{\rho_{31}^3}\right) = 0 \rightarrow (12)$$

since $\vec{r_1} \times \vec{r_2} \neq 0$.

$\therefore$ $\vec{r_1}$ is not parallel to $\vec{r_2}$.

$\therefore$ $\vec{r_3}$ canot be parallel to both $\vec{r_2}$ & $\vec{r_1}$ simulta-neously suppose that $\vec{r_3}$ is not parallel to $\vec{r_2}$

$$\therefore \vec{r_2} \times \vec{r_3} \neq 0$$

$$(9) \Rightarrow \rho_{23} = \rho_{12}$$

$$\therefore \rho_{23} = \rho_{12} = \rho_{31} = \rho \text{ (say)}$$

(5) $\Rightarrow$ $-n^2 \vec{r_1} = \frac{k^2}{\rho^3} [m_2(\vec{r_2} - \vec{r_1}) - m_3(\vec{r_1} - \vec{r_3})]$

$$= \frac{k^2}{\rho^3} [m_2 \vec{r_2} + m_3 \vec{r_3} - m_2 \vec{r_1} - m_3 \vec{r_1}]$$

$$= \frac{k^2}{\rho^3} [(m_1 \vec{r_1} + m_2 \vec{r_2} + m_3 \vec{r_3}) - (m_1 + m_2 + m_3)\vec{r_1}]$$

$$= -\frac{k^2}{\rho^3} (m_1 + m_2 + m_3) \vec{r_1}$$

$$\Rightarrow n^2 = \frac{k^2}{\rho^3} (m_1 + m_2 + m_3)$$

$$= \frac{k^2}{\rho^3} M, \text{ where } M = m_1 + m_2 + m_3$$

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$\Rightarrow \rho = \left(\frac{k^2 M}{n^2}\right)^{1/3}$, which gives the distance of the masses from the other.

Hence, in this case, the problem is solvable.

Here the distance of each pair of masses are same as they form an equitarial triangle and this solution is called equitaterial triangle solution of the three body problem.

Case (2):