For this problem we have,

$$u=0,\ \frac{\partial u}{\partial y}\neq 0,\ \frac{\partial^2 u}{\partial y^2}=0,\ y=0$$

$$u=U,\ \frac{\partial u}{\partial y}=0 \quad \text{at } y=\delta$$

$$a=c=0,\ b+d=1,\ b+3d=0$$

$$\therefore\ u=U\left(\tfrac{3}{2}\eta-\tfrac{1}{2}\eta^3\right)$$

$$\delta_1=\frac{3}{8}\delta,\quad \delta_2=\frac{39}{280}\delta$$

$$\therefore\ \frac{\sigma_0}{\rho}=\nu\frac{U}{\delta}\frac{3}{2},\quad \delta=\sqrt{\frac{280}{13}\frac{\nu}{U}x}$$

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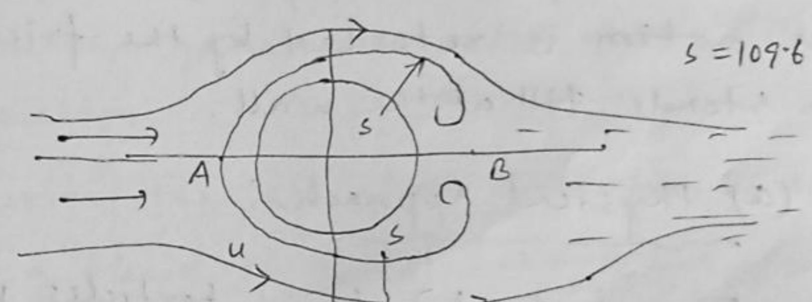
§: Boundary Layer equation for flow past a curved wall.



To establish the equation for 2D flow along a curved surfaces as in flow past a cylinderical obstacle, we use curvilinear orthogonal coordinates (α, β, γ). The motion is B.L. is two dimensional. So we may take γ as the coordinate z at right angle to the plane of motion, then the velocity component w is zero and all quantities are independent of γ. For curves $\alpha = \text{constant}$. We take normals to the wall and for the curves $\beta = \text{constant}$. We take the curves parallel to the wall each of which intersects the normals at a constant distance from the wall. The α is the distance measured along the wall from a fixed point which, for flow past a cylinder is taken as the forward stagnation point, while β is the normal distance from the wall.

It will cause no confusion and will allow us to treat plane and curved wall together, if we use x and y for α and β so that, quite generally, x and y are the distances along and perpendicular to the wall.

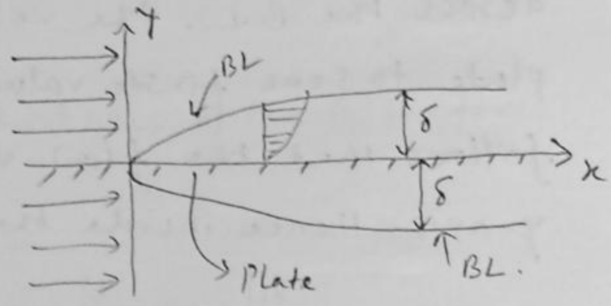
§: The seperation of Boundary wall:

When a fluid of small viscosity flows over a wall, a thin B.L. is developed along the wall, with infact, the inertia and viscous forces are of equal importance. In the B.L, the motion is retarded by the frictional forces and comes to stands till at the wall.

(a) Physical Appraach:

The declarated fluid particles in the B.L donot, in all cases, remain in the thin layer which, adheres to the body along the ~~wot~~ whole wetted length of the wall. In some, cases, the B.L. increases its thickness considerably in the down stream direction. The causes the declarated fluid particles to be forced ~~outside~~ outwards which means that the B.L. is separated from the wall. We then speak of B.L. seperation. The point at which the B.L. separates is called the point of separation. This phenomena is always associated with the formation of vortices and with large energy losses in the wake of the body. It occurs primarily near blunt bodies such as circular cylinders, spheres etc. A consequence of B.L. separation is that the body experiences a large drag.

§: Boundary Layer equation about a flate plate at zero incidence:



The approximation resulting from the concept of the B.L. may be called B.L. approximation and the equations obtained by using this approximation in the general equation of motion are called the B.L. equation.

Let us now derive the equation of motion so called B.L. equation for a fluid in a laminar B.L.

For the sake of simplicity, we consider a 2D flow along a thin plate at $y=0$ $(x>0)$ as shown in figure. The undisturbed flow is in the direction of the x-axis. We suppose that a thin B.L. is developed on the plate. This is in the sense that ~~the~~ B.L. thickness δ is very small compared ~~the~~ a linear dimension L (say).

Now the exact Navier-Stokes equation of motion and the equation of continuity are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \longrightarrow (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \longrightarrow (ii)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (iii)$$

Here u and v are velocity components in the directions of x and y respectively, ρ is the density, $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. Since the B.L. developed is very thin and since across the B.L. the velocity changes from the value zero at the plate to some finite value outside the B.L., therefore, it follows that the flow velocity changes rapidly parallel to the y-axis. Hence inside the B.L.

$$\frac{\partial u}{\partial y} \sim \frac{U}{\delta}$$

where U is the main stream velocity and also

$$\frac{\partial^2 u}{\partial y^2} \sim \frac{U}{\delta^2}$$

Again, parallel to x-axis, the flow velocity varies slowly, and appropriate change in it occur only over a distance of order L. Hence,

$$\frac{\partial u}{\partial x} \sim \frac{U}{L}, \quad \frac{\partial^2 U}{\partial x^2} \sim \frac{U}{L^2}$$

We assume that $\frac{U}{L}$ is of magnitude unity and so $\frac{\partial u}{\partial x} \sim 1$.

Also, we assume $t \sim 1$.

Now, by equation (iii)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial y} \sim \frac{\partial u}{\partial x}$$

i.e. $\frac{\partial v}{\partial y} \sim 1$

i.e. $v \sim \delta$.

and so, $\frac{v}{U} \sim \frac{\delta}{U}$.

This show that $v << u$,
which means inside the B.L., the flow takes place nearly parallel to the plate.

Again, $\frac{\partial^2 v}{\partial y^2} \sim \frac{\delta}{\delta^2} \sim \frac{1}{\delta}$

and $\frac{\partial v}{\partial x} \sim \frac{\delta}{L}$

so $\frac{\partial^2 v}{\partial x^2} \sim \frac{\delta}{L^2}$

Hence as a consequence of the thickness of the B.L. for any flow quantity

$$\frac{\partial}{\partial x} \leq \frac{\partial}{\partial y}$$

also, $\frac{\partial^2}{\partial x^2} << \frac{\partial^2}{\partial y^2}$

Thus, at points inside the B.L., we may use the approximation

$$\left|\frac{\partial u}{\partial x}\right| << \left|\frac{\partial u}{\partial y}\right| \quad , \quad \left|\frac{\partial^2 u}{\partial x^2}\right| \leq \left|\frac{\partial^2 u}{\partial y^2}\right|$$

with this approximation, equation (1) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \longrightarrow (IV)$$

Hence, the inertia term $\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)$ and the viscous term $\left(\nu\frac{\partial^2 u}{\partial y^2}\right)$ are supposed to be of same order U.

As regards equation (II), we find that each of the term in the L.S. of equation (II) is of order δ.

$$\left|\frac{\partial^2 v}{\partial x^2}\right| << \left|\frac{\partial^2 v}{\partial y^2}\right|$$

and $\nu\frac{\partial^2 v}{\partial y^2} \sim \delta$.

I.T. $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$

$\sim U\frac{U}{L} + \delta \cdot \frac{U}{\delta}$

$\sim U + U$

$\sim U$

$$\frac{\partial v}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\quad)$$

Hence with the neglect of term of order of magnitude δ, we have the approximate relation from (II) as

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \longrightarrow \text{(V)}$$

But the equation (III) remains, unaffected by this approximation. Equation (V) shows that the pressure is approximately uniform across the B.L. Hence the pressure in the B.L. may be taken to be equal to the pressure outside the B.L. in the main stream. But for the pressure in the main stream, where the influence of viscosity is negligible, we may write

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 0$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \longrightarrow \text{(VI)}$$

In steady state (VI) becomes,

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \longrightarrow \text{(VII)}$$

$$\text{or, } \frac{1}{2}\cdot 2U\frac{dU}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\text{or, } \frac{1}{2}\frac{dU^2}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\text{or, } \frac{1}{2}\int \frac{dU^2}{dx}\,dx = -\frac{1}{\rho}\int \frac{\partial p}{\partial x}\,dx$$

$$\text{or, } \frac{1}{2}U^2 = -\frac{1}{\rho}p + \text{constant.}$$

which is equivalent to Bernoulli's equation

$$\frac{1}{2}U^2 + \frac{p}{\rho} = \text{constant.}$$

Hence, in steady case using (VII) in (IV), we have,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

Thus the steady two-dimension B.L. equation are

$$\left.\begin{aligned} u\frac{du}{dx} + v\frac{du}{dy} &= U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \longrightarrow \text{(VIII)}$$

Equation (VIII) are also called Prandtl's B.L. equation or simplified Navier Stoke's equation in 2D B.L. theory.

The B.Condition related to the equation (VIII) are

$$u = v = 0 \quad \text{at } y = 0$$

$$u = U(x) \quad \text{at } y = \delta.$$

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