Unit 2

Riemann - christoffel Curvature tensor :

Def^n :- Let A_i be a covariant vector. If x^j - covariant derivative given by —

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \Gamma_{ij}^{\alpha} A_\alpha \longrightarrow (1)$$

is a covariant tensor of rank two.

$\therefore$ x^k - covariant derivative of $A_{i,j}$ is given by —

$$A_{i,jk} = \frac{\partial A_{i,j}}{\partial x^k} - \Gamma_{jk}^{\alpha} A_{i,\alpha} - \Gamma_{ik}^{\alpha} A_{\alpha,j} \longrightarrow (2)$$

Interchanging j and k in (2), we get —

$$A_{i,kj} = \frac{\partial A_{i,k}}{\partial x^j} - \Gamma_{kj}^{\alpha} A_{i,\alpha} - \Gamma_{ij}^{\alpha} A_{\alpha,k} \longrightarrow (3)$$

$$\because \Gamma_{jk}^{\alpha} = \Gamma_{kj}^{\alpha}$$

$$\therefore (2)-(3) \Rightarrow A_{i,jk} - A_{i,kj} = \frac{\partial A_{i,j}}{\partial x^k} - \frac{\partial A_{i,k}}{\partial x^j} - \Gamma_{ik}^{\alpha} A_{\alpha,j} + \Gamma_{ij}^{\alpha} A_{\alpha,k}$$

Using (1), we get —

$$A_{i,jk} - A_{i,kj} = \frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^j} - \Gamma_{ij}^{\alpha} A_\alpha\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial A_i}{\partial x^k} - \Gamma_{ik}^{\alpha} A_\alpha\right) -$$

$$\Gamma_{ik}^{\alpha}\left(\frac{\partial A_\alpha}{\partial x^j} - \Gamma_{\alpha j}^{\beta} A_\beta\right) + \Gamma_{ij}^{\alpha}\left(\frac{\partial A_\alpha}{\partial x^k} - \Gamma_{\alpha k}^{\beta} A_\beta\right)$$

$$= \frac{\partial^2 A_i}{\partial x^k \partial x^j} - \Gamma_{ij}^{\alpha} \frac{\partial A_\alpha}{\partial x^k} - \frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} A_\alpha - \frac{\partial^2 A_i}{\partial x^j \partial x^k} + \Gamma_{ik}^{\alpha} \frac{\partial A_\alpha}{\partial x^j}$$

$$\partial \Gamma^{\alpha}$$

$$\therefore A_{i,jk} - A_{i,kj} = A_\alpha\left(-\frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} + \frac{\partial \Gamma_{ik}^{\alpha}}{\partial x^j} - \Gamma_{ij}^{\beta} \Gamma_{\beta k}^{\alpha} + \Gamma_{ik}^{\beta} \Gamma_{\beta j}^{\alpha}\right) \longrightarrow (4)$$

The R.H.S. of (4) is a covariant tenson of rank three and A_α on the R.H.S. is an arbitrary covariant vector. Hence by Quotient theorem of tensors the expression —

$$-\frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} + \frac{\partial \Gamma_{ik}^{\alpha}}{\partial x^j} - \Gamma_{ij}^{\beta} \Gamma_{\beta k}^{\alpha} + \Gamma_{ik}^{\beta} \Gamma_{\beta j}^{\alpha}$$

on the R.H.S. must be a mixed tensor of rank four, contravariant in α and covariant in i, j, k

This tensor is called the Riemann - christoffel curvature tensor or the Riemann tensor or Riemann's symbol of the second kind. It is denoted by R_{ijk}^{α}. Thus —

$$R_{ijk}^{\alpha} = -\frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} + \frac{\partial \Gamma_{ik}^{\alpha}}{\partial x^j} - \Gamma_{ij}^{\beta} \Gamma_{\beta k}^{\alpha} + \Gamma_{ik}^{\beta} \Gamma_{\beta j}^{\alpha}$$

Note : $(4) \Rightarrow A_{i,jk} - A_{i,kj}$

$\because A_{i,jk} = A_{i,kj} \Leftrightarrow R^{\alpha}_{ijk} = 0$

Properties :

1. Antisymmetric Property : $R^{\alpha}_{ijk} = -R^{\alpha}_{ikj}$

<u>Proof :</u> We have —

$$R^{\alpha}_{ijk} = -\frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^k} + \frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^j} - \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k} + \Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j}$$

Interchanging j and k, we get —

$$R^{\alpha}_{ikj} = -\frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^j} + \frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^k} - \Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta k} + \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k}$$

$$= -\left(\frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^k} + \frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^j} - \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k} + \Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta k} \right)$$

$$= -R^{\alpha}_{ijk}$$

$\therefore R^{\alpha}_{ijk} = -R^{\alpha}_{ikj}$

2. Cyclic Property : $R^{\alpha}_{ijk} + R^{\alpha}_{jki} + R^{\alpha}_{kij} = 0$

<u>Proof :</u> We have —

$$R^{\alpha}_{ijk} = -\frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^k} - \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k} + \Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j}$$

Rotating the indices i, j, k cyclically, we get —

$$R^{\alpha}_{jki} = -\frac{\partial \Gamma^{\alpha}_{jk}}{\partial x^i} + \frac{\partial \Gamma^{\alpha}_{ji}}{\partial x^k} - \Gamma^{\beta}_{jk}\Gamma^{\alpha}_{\beta i} + \Gamma^{\beta}_{ji}\Gamma^{\alpha}_{\beta k}$$

$$R^{\alpha}_{kij} = -\frac{\partial \Gamma^{\alpha}_{ki}}{\partial x^j} + \frac{\partial \Gamma^{\alpha}_{kj}}{\partial x^i} - \Gamma^{\beta}_{ki}\Gamma^{\alpha}_{\beta j} + \Gamma^{\beta}_{kj}\Gamma^{\alpha}_{\beta i}$$

Adding and using the symmetric property of christoffel symbol of 2nd kind we get —

$$R^{\alpha}_{ijk} + R^{\alpha}_{jki} + R^{\alpha}_{kij} = 0$$