Thus the steady two-dimension B.L. equation are

$$\left.\begin{aligned} u\frac{du}{dx}+v\frac{du}{dy} &= U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \longrightarrow (\text{viii})$$

Equation (viii) are also called Prandtl's B.L. equation or simplified Navier Stoke's equation in 2D B.L. theory.

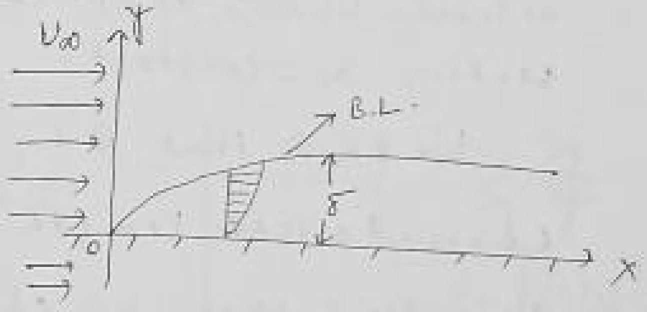
The B.Condition related to the equation (viii) are

$$u = v = 0 \quad \text{at } y = 0$$

$$u = U(x) \quad \text{at } y = \delta .$$

#

6. Similarity solution of B.L. Equations:



The solution of B.L. equation was given by Blasius (1908).

Let the plane of the thin plate be the xz half plate with $x>0$, the leading edge of the plate be at $y=0$. We suppose that the plate to extend indefinitely in the x-direction.

Now, we consider a steady viscous fluid flow with a free stream velocity U_∞ which is parallel to the x-axis past the plate. We suppose that a thin B.L. is developed on either side of the plate as shown in figure. Since the plate is assumed to be very thin it causes no obstraction to the flow except producing some effect of the B.L. developed on it. But as the B.L. is supposed to be thin, its effects on the main stream is inapplicable and so velocity of the main stream can be taken as constant being equal to U_∞.

The steady flow inside the B.L. governed by the B.L. equations are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \longrightarrow (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (2)$$

Subjecting boundary condition

$$\left.\begin{array}{ll} u = v = 0 & \text{at } y = 0 \\ U = U_\infty(x) & \text{at } y \to \infty \end{array}\right\} \to (3)$$

where u, v are the velocity components in the x and y direction respectively and $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity.

By a similarity or similar solution we mean a solution which shows that velocity distribution for u at varying section x_0 = constant.

We know that B.L. thickness $\delta \sim \sqrt{\frac{\nu L}{U_\infty}}$ if L is the characteristic length. But since the plate is infinite, to extend there no characteristic length L is available. So, we define the B.L. thickness as

$$\delta(x) \sim \sqrt{\frac{\nu x}{U_\infty}}$$

We define,

$u = U_\infty \, \phi\left(\frac{y}{\delta}\right)$, where ϕ is the dimensionless function and if ϕ is found to satisfy some differential equation which does not involve x and y explicitly, then the distribution of u will remain the same accross varying sections x = constant.

Thus for similar solution we introduce the dimensionless coordinate η such that

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \longrightarrow (4)$$

In view of the continuity equation (2), a stream function ψ can be defined such that

$$u = -\frac{\partial\psi}{\partial y},\quad v = \frac{\partial\psi}{\partial x} \longrightarrow (5)$$

We now put $\psi = -\sqrt{U_\infty \nu x}\, f(\eta)$

Then $u = -\frac{\partial\psi}{\partial y} = U_\infty f'(\eta)$

$$v = \frac{\partial\psi}{\partial x} = \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}\left\{\eta f' - f\right\}.$$

where the prime denotes differential w.r. to η.

Now, $\frac{\partial u}{\partial x} = U_\infty f''(\eta)\frac{\partial\eta}{\partial x}$

$$= U_\infty f''(\eta)\sqrt{\frac{U_\infty}{\nu}}\; y\left(-\frac{1}{2}\right)\frac{1}{x\sqrt{x}} \;;$$

$$= -\frac{U_\infty}{2x}\,\eta f''(\eta)$$

$$\frac{\partial v}{\partial y} = U_\infty f''(\eta)\sqrt{\frac{U_\infty}{\nu x}}$$

and $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[U_\infty f''(\eta)\sqrt{\frac{U_\infty}{\nu x}}\right]$

$$= \frac{U_\infty^2}{\nu x} f'''(\eta).$$

$$u = -\frac{\partial\psi}{\partial y} = -\frac{\partial\psi}{\partial\eta}\cdot\frac{\partial\eta}{\partial y}$$

$$= \sqrt{U_\infty \nu x}\, f'(\eta)\sqrt{\frac{U_\infty}{\nu x}}$$

$$= U_\infty f'(\eta)$$

$$\eta = \frac{y}{\delta} = y\sqrt{\frac{U_\infty}{\nu x}}$$

Now, substituting the values of $u, v, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial y^2}$ etc. in (1) we get,

$$ff'' + 2f''' = 0 \longrightarrow (7)$$

Thus the P.D. equation (1) is transformed to the ordinary differential equation. The equation (7) does not involve x & y and hence this confirms the existence of similarity condition.

The boundary condition for f are:

$$\left.\begin{aligned} f = f' = 0 \quad &\text{at } \eta = 0 \\ f' = 1 \quad &\text{at } \eta \to \infty \end{aligned}\right\} \to (8)$$

The differential equation (7) is to be solved subject to the boundary condition (8). Thus differential equation is non-linear and can be solved numerically. However for small values of η i.e. around $\eta \to 0$. H. Blasius obtained this set in the form of a series expansion and asymptote expansion for η very large. The two forms of set: are to be matched at a suitable value of η. The graph of $f(\eta)$ which gives the velocity distribution i.e. the distribution of $\frac{U}{U_\infty}$ across any section of the B.L. is shown below.

It is observed that f rises rapidly and then asymptotically to the limiting value of units. It may be noted that at the outer edge of the B.L. i.e. for $\eta \to \infty$. the transverse component of velocity differs from zero.

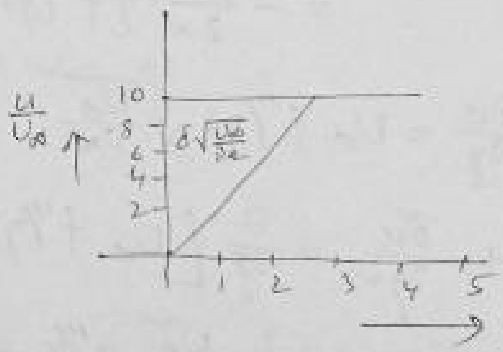


Fig. Vel. distribution in the B.L along a flat after Blasius.

We have,

$$v_\infty = \cancel{0.68}\ 0.8604\, U_\infty \sqrt{\frac{\nu}{x U_\infty}}$$

Another important point to be observed is that the set: for u & v as obtained above are valid for $\frac{U_\infty x}{\nu} >> 1$. In the neighbourhood of the leading edge at $x=0$, the B.L. approximation fails and consequently the set (solⁿ) are not valid therefore the important improved approximations are necessary [L. Rocesslard].

Skin Friction:

The frictional force per unit area of the plate at distance x from the leading edge is called shearing stress at the wall.

In the case of flow past a ~~flate~~ plate

$$\sigma_W = (\sigma_{xy})_{y=0} = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu U_\infty f''(0)\sqrt{\frac{U_\infty}{\nu x}}$$

$$\sigma_W = \mu U_\infty \alpha \sqrt{\frac{U_\infty}{\nu x}}, \quad \text{where } \alpha = f''(0) = 0.332$$

$$= \rho \nu U_\infty \alpha \frac{1}{\sqrt{\frac{\nu x}{U_\infty} \times \frac{U_\infty^2}{\nu^2}}} \alpha \frac{U_\infty}{\nu}$$

$$= \rho U_\infty^2 \alpha \frac{1}{\sqrt{\frac{U_\infty x}{\nu}}} = \rho U_\infty^2 (0.332) \frac{1}{\sqrt{R_L}}, \quad \text{where } R_L \text{ is the Reynolds no.}$$

Hence the dimensionless shearing stress which is also known as local skin friction coefficient is given by

$$C_f = \frac{\sigma_W}{\rho U_\infty^2/2} = \frac{0.664}{\sqrt{R_L}}$$

The drag extend on the two sides of a plate of length l width b is given by

$$D = 2b \int_{x=0}^{l} (\sigma_{xy})_{y=0}\, dx$$