Riemann-Christoffel curvature tensor:

Definition: Let A_i be a covariant vector. Its x^j-covariant derivative given by

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \Gamma^{\alpha}_{ij} A_\alpha \longrightarrow (1)$$

is a covariant tensor of rank two.

$\therefore$ x^k-covariant derivative of $A_{i,j}$ is given by

$$A_{i,jk} = \frac{\partial A_{i,j}}{\partial x^k} - \Gamma^{\alpha}_{jk} A_{i,\alpha} - \Gamma^{\alpha}_{ik} A_{\alpha,j} \longrightarrow (2)$$

$\boxed{(A_{i,j})_{,k} = A_{i,jk}}$

Interchanging j and k in (2), we get

$$A_{i,kj} = \frac{\partial A_{i,k}}{\partial x^j} - \Gamma^{\alpha}_{kj} A_{i,\alpha} - \Gamma^{\alpha}_{ij} A_{\alpha,k} \longrightarrow (3)$$

$\because \Gamma^{\alpha}_{jk} = \Gamma^{\alpha}_{kj}$

$\therefore$ (2) $-$ (3) $\Rightarrow$

$$A_{i,jk} - A_{i,kj} = \frac{\partial A_{i,j}}{\partial x^k} - \frac{\partial A_{i,k}}{\partial x^j} - \Gamma^{\alpha}_{ik} A_{\alpha,j} + \Gamma^{\alpha}_{ij} A_{\alpha,k}$$

Using (1), we get

$$A_{i,jk} - A_{i,kj} = \frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^j} - \Gamma^{\alpha}_{ij} A_\alpha\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial A_i}{\partial x^k} - \Gamma^{\alpha}_{ik} A_\alpha\right) - \Gamma^{\alpha}_{ik}\left(\frac{\partial A_\alpha}{\partial x^j} - \Gamma^{\beta}_{\alpha j} A_\beta\right) + \Gamma^{\alpha}_{ij}\left(\frac{\partial A_\alpha}{\partial x^k} - \Gamma^{\beta}_{\alpha k} A_\beta\right)$$

$$= \frac{\partial^2 A_i}{\partial x^k \partial x^j} - \Gamma^{\alpha}_{ij}\frac{\partial A_\alpha}{\partial x^k} - \frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^k} A_\alpha - \frac{\partial^2 A_i}{\partial x^j \partial x^k} + \Gamma^{\alpha}_{ik}\frac{\partial A_\alpha}{\partial x^j} + \frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^j} A_\alpha - \Gamma^{\alpha}_{ik}\frac{\partial A_\alpha}{\partial x^j} + \Gamma^{\alpha}_{ik}\Gamma^{\beta}_{\alpha j} A_\beta + \Gamma^{\alpha}_{ij}\frac{\partial A_\alpha}{\partial x^k} - \Gamma^{\alpha}_{ij}\Gamma^{\beta}_{\alpha k} A_\beta$$

$$\therefore A_{i,jk} - A_{i,kj} = A_\alpha \left(-\frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} + \frac{\partial \Gamma_{ik}^{\alpha}}{\partial x^j} - \Gamma_{ij}^{\beta}\Gamma_{\beta k}^{\alpha} + \Gamma_{ik}^{\beta}\Gamma_{\beta j}^{\alpha} \right) \longrightarrow (4)$$

The ~~R~~L.H.S of (4) is a covariant tensor of rank three and A_α on the R.H.S is an arbitrary covariant vector. Hence by quotient theorem of tensors the expression

$$-\frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} + \frac{\partial \Gamma_{ik}^{\alpha}}{\partial x^j} - \Gamma_{ij}^{\beta}\Gamma_{\beta k}^{\alpha} + \Gamma_{ik}^{\beta}\Gamma_{\beta j}^{\alpha}$$

on the R.H.S must be a mixed tensor of rank four contravariant in α and covariant in i, j, k.

This tensor is called Riemann christoffell curvature tensor or the Riemann tensor, or Riemann's symbol of the second kind. It is denoted by R^{α}_{ijk}.

Thus $$R^{\alpha}_{ijk} = -\frac{\partial \Gamma_{ij}^{\alpha}}{\partial x^k} + \frac{\partial \Gamma_{ik}^{\alpha}}{\partial x^j} - \Gamma_{ij}^{\beta}\Gamma_{\beta k}^{\alpha} + \Gamma_{ik}^{\beta}\Gamma_{\beta j}^{\alpha}$$

Note: $A_{i,jk} - A_{i,kj} = A_\alpha R^{\alpha}_{ijk}$ [from (4)]

$$\therefore A_{i,jk} = A_{i,kj} \Leftrightarrow R^{\alpha}_{ijk} = 0$$

Scanned with CamScanner