Series solution for η large (Asymptotic solution)

The displacement thickness is defined by (in this problem)

$$U_\infty \delta_1 = \int_{y=0}^{\infty} (U_\infty - u)\, dy$$

when $\eta = \left(\frac{U_\infty}{\nu x}\right)^{\frac{1}{2}} y$ and $u = U_\infty f'(\eta)$

$$\delta_1 = \int_{y=0}^{\infty} \left(1 - \frac{u}{U_\infty}\right) dy$$

$$= \int_{\eta=0}^{\infty} (1-f') \left(\frac{\nu x}{U_\infty}\right)^{\frac{1}{2}} d\eta$$

$$= \left(\frac{\nu x}{U_\infty}\right)^{\frac{1}{2}} \Big[(\eta - f)\Big]_0^{\infty} = \left(\frac{\nu x}{U_\infty}\right)^{\frac{1}{2}} \lim_{\eta\to\infty} (\eta - f)$$

This shows that when η is very large, $(\eta - f)$ tends to a definite value β (say) then

$$\delta_1 = \left(\frac{\nu x}{U_\infty}\right)^{\frac{1}{2}} \beta$$

Hence for η large, we can write,

$f = \eta - \beta + \phi(\eta)$, where $\phi(\eta)$ is small and tends to zero as $\eta \to \infty$.

Substituting this in Blasius equation $ff'' + 2f''' = 0$ we get,

$$2\phi''' + (\eta - \beta)\phi'' = 0 \quad \text{(neglecting } \phi\phi'' \text{ as it is small)}$$

or, $2\phi''' + \xi \phi'' = 0$, where $\xi = \eta - \beta$

Integrating, $\phi'' \sim A \exp\left(-\frac{\xi^2}{4}\right)$ and that

$$\phi \sim A\, \xi^{-2} \exp\left(-\frac{\xi^2}{4}\right).$$

Thus for approximate formula for $f(\eta)$ valid for η large is

$$f(\eta) \approx (\eta-\beta) + A(\eta-\beta)^{-2} \exp\left\{-\frac{1}{4}(\eta-\beta)^2\right\}$$

where A is some constant to be determined by matching this solution with that of η small at some suitable value of η.

Blasius obtained the numerical solution of $ff'' + 2f''' = 0$ and from that solution it is found that

$$\alpha = f''(0) = 0.3321$$

$$\beta = 1.7208$$

and with their values of α and β, the constant A can be determined and it is found that

$$A = 0.231.$$

[Skin friction → similar as that obtained in similarity solⁿ.]

§. Kármán's momentum integral equation:

For two dimensional flow of incompressible fluid over a semi-infinite flat plate, Prandtl's B.L. equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \longrightarrow (i)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (ii)$$

subject to boundary condition

$$\left.\begin{aligned} u = v = 0 &\quad \text{at } y = 0 \\ u = U(x,t) &\quad \text{at } y = \delta \end{aligned}\right\} \longrightarrow (iii)$$

But for the pressure in the main stream, where influence of viscosity is negligible, we may write,

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \longrightarrow (\text{IV})$$

and $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{\partial u}{\partial y}$$

$$= 2u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

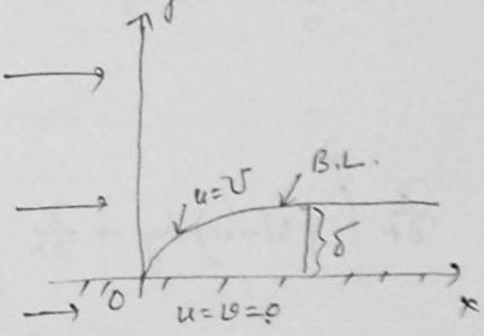
$$= \frac{\partial u^2}{\partial x} + \frac{\partial}{\partial y}(uv) \longrightarrow (\text{V})$$

Now, using (V), (I) becomes,

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial}{\partial y}(uv) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \longrightarrow (\text{VI})$$

Integrating (VI) w.r. to 'y' across the B.L.

$$\int_{y=0}^{\delta}\frac{\partial u}{\partial t}dy + \int_0^{\delta}\frac{\partial u^2}{\partial x}dy + \int_0^{\delta}\frac{\partial}{\partial y}(uv)dy = -\frac{1}{\rho}\int_0^{\delta}\frac{\partial p}{\partial x}dy + \int_0^{\delta}\frac{\partial^2 u}{\partial y^2}dy.$$



$$\Rightarrow \frac{\partial}{\partial t}\int_0^{\delta}u\,dy + \frac{\partial}{\partial x}\int_0^{\delta}u^2dy + \left[uv\right]_{y=0}^{\delta} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\left[y\right]_{y=0}^{\delta}$$

$$+\nu\left[\frac{\partial u}{\partial y}\right]_{y=0}^{\delta}$$

$$\Rightarrow \frac{\partial}{\partial t}\int_0^{\delta}u\,dy + \frac{\partial}{\partial x}\int_0^{\delta}u^2dy + uV = -\frac{\delta}{\rho}\frac{\partial p}{\partial x} - \nu\left(\frac{\partial u}{\partial y}\right)_{y=0}.$$

$$\Rightarrow \frac{\partial}{\partial t}\int_0^{\delta}u\,dy + \frac{\partial}{\partial x}\int_0^{\delta}u^2dy - V\int_0^{\delta}\frac{\partial u}{\partial x}dy = -\frac{\delta}{\rho}\frac{\partial p}{\partial x} - \nu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= -\frac{\delta}{\rho}\frac{\partial p}{\partial x} - \frac{\sigma_0}{\rho}$$

where $\sigma_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is the shearing

$u = v = 0$ at $y = 0$

$\frac{\partial u}{\partial y} = 0$ at $y = 0$

$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0$ at $y = \delta$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$\Rightarrow \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

$\therefore V = -\int\frac{\partial u}{\partial x}dy$

$\int_0^{\delta}dy = [y]_0^{\delta} = \delta$

stress at the plate.

Again, by (IV)

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}$$

i.e. $$-\frac{\delta}{\rho}\frac{\partial p}{\partial x} = \delta\frac{\partial U}{\partial t} + U\delta\frac{\partial U}{\partial x}$$

$$= \frac{\partial}{\partial t}\int_0^\delta U\,dy + \int_0^\delta U\frac{\partial U}{\partial x}\,dy \longrightarrow \text{(VIII)}$$

Now, using (VIII), (VII) becomes,

$$\frac{\partial}{\partial t}\int_0^\delta u\,dy + \frac{\partial}{\partial x}\int_0^\delta u^2\,dy - U\int_0^\delta \frac{\partial u}{\partial x}\,dy = \frac{\partial}{\partial t}\int_0^\delta U\,dy + \int_0^\delta U\frac{\partial V}{\partial x}\,dy = \frac{\sigma_0}{\rho}.$$

i.e. $$\frac{\partial}{\partial t}\int_0^\delta (U-u)\,dy - \frac{\partial}{\partial x}\int_0^\delta u^2\,dy + \frac{\partial}{\partial x}\int_0^\delta Uu\,dy - \frac{\partial u}{\partial x}\int_0^\delta u\,dy + \frac{\partial U}{\partial x}\int_0^\delta U\,dy = \frac{\sigma_0}{\rho}$$

i.e. $$\frac{\partial}{\partial t}\int_0^\delta (U-u)\,dy + \frac{\partial}{\partial x}\int_0^\delta u(U-u)\,dy + \frac{\partial u}{\partial x}\int_0^\delta (U-u)\,dy = \frac{\sigma_0}{\rho},$$

where $\delta \to \infty$, then we have,

$$\frac{\partial}{\partial t}\int_0^\infty (U-u)\,dy + \frac{\partial}{\partial x}\int_0^\alpha U(U-u)\,dy + \frac{\partial U}{\partial x}\int_0^\alpha (U-u)\,dy = \frac{\sigma_0}{\rho}.$$

i.e. $$\frac{\partial}{\partial t}(U\delta_1) + \frac{\partial}{\partial x}(U^2\delta_2) + U\delta_1\frac{\partial U}{\partial x} \pm \frac{\sigma_0}{\rho} \longrightarrow \text{(IX)}$$

where $\delta_1 = \int_0^\infty \left(1-\frac{u}{U}\right)dy$ is the displacement thickness

and $\delta_2 = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy$ is the momentum thickness.

The equation (IX) is called Karman's momentum integral equation (1921).

For steady flow, the equation (IX) becomes,

$$\frac{\partial}{\partial x}(U^2\delta_2) + U\delta_1\frac{\partial U}{\partial x} = \frac{\sigma_0}{\rho}$$

$$\frac{\partial}{\partial x}\int_0^\infty u(U-u)\,dy + \frac{dU}{dx}\int_0^\infty (U-u)\,dy = \frac{\sigma_0}{\rho} \longrightarrow (x)$$

For steady flow, under no pressure gradient, the equation (x) becomes,

$$\frac{\partial}{\partial x}\int_0^\infty u(U-u)\,dy = \frac{\sigma_0}{\rho}$$

or, $$\sigma_0 = \rho\,\frac{\partial}{\partial x}\int_0^\infty u(U-u)\,dy\,.$$

x