Ordinary and Singular points:

Def^n- A point $x=x_0$ is called an ordinary point of the equation $y'' + P(x)y' + Q(x)y = 0 \rightarrow (1)$ if both the function $P(x)$ and $Q(x)$ are analytic at $x=x_0$.

If the point $x=x_0$ is not an ordinary point of the differential eq^n (1), then it is called a singular point of the differential eq^n (1), there are two types of singular points (i) regular singular points, (ii) irregular singular points.

A singular point $x=x_0$ of the differential eq^n (1) is called a regular singular point of (1) if both $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ are analytic at $x=x_0$.

A singular point which is not regular is called irregular singular point.

Ex Show that $x=0$ is an ordinary point of $(x^2-1)y'' + xy' - y = 0$ but $x=1$ is a regular singular point.

Solve: The given eqn is

$$y'' + \frac{x}{x^2-1} y' - \frac{1}{(x+1)(x-1)} y = 0 \quad \text{—(1)}$$

Comparing (1) with $y'' + P(x)y' + Q(x)y = 0$

, $P(x) = \dfrac{x}{(x+1)(x-1)}$, $Q(x) = \dfrac{-1}{(x+1)(x-1)}$

Since both $P(x)$ and $Q(x)$ are ~~not~~ analytic at $x=0$, so $x=0$ is an ordinary point

Since both $P(x)$ and $Q(x)$ are undefined at $x=1$, so they are not analytic at $x=1$ and thus $x=1$ is not an ordinary point and so $x=1$ is a singular point.

Also $(x-1)P(x) = \dfrac{x}{x+1}$, and $(x-1)^2 Q(x) = \dfrac{x-1}{x+1}$

Thus both $(x-1)P(x)$ and $(x-1)^2 Q(x)$ are ~~at~~ analytic at $x=1$ therefore $x=1$ is a regular point

(11) Ex. Determine whether $x=0$ is an ordinary point or a regular singular point of the differential eqn

$2x^2(d^2y/dx^2) + 7x(x+1)(dy/dx) - 3y = 0$

Soln Dividing by $2x^2$, the given eqn becomes

$$\frac{d^2y}{dx^2} + \frac{7(x+1)}{2x}\frac{dy}{dx} - \frac{3}{2x^2}y = 0 \quad \longrightarrow (1)$$

Comparing (1) with standard eqn

$y'' + P(x)y' + Q(x)y = 0$ we have

$$P(x) = \frac{7(x+1)}{2x} \quad \text{and} \quad Q(x) = \frac{-3}{2x^2} \quad \longrightarrow (2)$$

Since both $P(x)$ and $Q(x)$ are undefined at $x=0$, so both $P(x)$ and $Q(x)$ are not analytic at $x=0$ Thus $x=0$ is not an ordinary point So $x=0$ is a singular point.

Also $(x-0)P(x) = \frac{7(x+1)}{2}$

and $(x-0)^2 Q(x) = \frac{-3}{2}$

both $(x-0)P(x)$ and $(x-0)^2 Q(x)$ are analytic at $x=0$, therefore $x=0$ is a regular singular point.

Q3

Show that $x=0$ and $x=-1$ are singular points of $x^2(x+1)^2y'' + (x^2-1)y' + 2y = 0$, where the first is irregular and the other is regular.

Soln Dividing by $x^2(x+1)$, the given eqn becomes

$$\frac{d^2y}{dx^2} + \frac{x-1}{x^2(x+1)}\frac{dy}{dx} + \frac{2}{x^2(x+1)^2}y = 0$$

Comparing ① with standard eqn $y'' + P(x)y' + Qy = 0$

we get $P(x) = \frac{x-1}{x^2(1+x)}$ and $Q(x) = \frac{2}{x^2(x+1)^2}$

Since both $P(x)$ and $Q(x)$ are undefined at $x=0$ and $x=-1$, So they are not analytic at $x=0$ and $x=-1$, Hence $x=0$, and $x=-1$ are singular point.

~~Also~~ Also $(x-0)P(x) = \frac{x-1}{x(1+x)}$

and $(x-0)^2Q(x) = \frac{2}{(x+1)^2}$

Showing that $P(x)$ is not analytic at $x=0$

So x is a irregular singular point.

Again $(x+1)P(x) = \frac{x-1}{x^2}$

and $(x+1)^2Q(x) = \frac{2}{x^2}$

Showing that both $(x+1).P(x)$ and $(x+1)^2 Q(x)$ are analytic at $x=-1$ and hence $x=-1$ is a regular ~~analytic~~ singular point.