Properties:

1) Anti-symmetric property:

$$R^{\alpha}_{ijk} = - R^{\alpha}_{ikj}$$

proof: We have

$$R^{\alpha}_{ijk} = -\frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^{k}} + \frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^{j}} - \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k} + \Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j}$$

Interchanging j and k, we get

$$R^{\alpha}_{ikj} = -\frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^{j}} + \frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^{k}} - \Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j} + \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k}$$

$$\cancel{= \frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^{k}}}$$

$$= -\left(-\frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^{k}} + \frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^{k}} - \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k} + \Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j}\right)$$

$$= - R^{\alpha}_{ijk}$$

$$\therefore \quad R^{\alpha}_{ijk} = - R^{\alpha}_{ikj}$$

2) cyclic property:

$$R^{\alpha}_{ijk} + R^{\alpha}_{jki} + R^{\alpha}_{kij} = 0 .$$

We have

$$R^{\alpha}_{ijk} = -\frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^{k}} + \frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^{j}} - \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k} + \Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j} .$$

Rotating the indicies i, j, k cyclically, we get

$$R^{\alpha}_{jki} = -\frac{\partial \Gamma^{\alpha}_{jk}}{\partial x^{i}} + \frac{\partial \Gamma^{\alpha}_{ji}}{\partial x^{k}} - \Gamma^{\beta}_{jk}\Gamma^{\alpha}_{\beta i} + \Gamma^{\beta}_{ji}\Gamma^{\alpha}_{\beta k}$$

$$R^{\alpha}_{kij} = -\frac{\partial \Gamma^{\alpha}_{ki}}{\partial x^{j}} + \frac{\partial \Gamma^{\alpha}_{kj}}{\partial x^{i}} - \Gamma^{\beta}_{ki}\Gamma^{\alpha}_{\beta j} + \Gamma^{\beta}_{kj}\Gamma^{\alpha}_{\beta i}$$

Scanned with CamScanner

Adding and using the symmetry property of christoffel symbols of the second kind we get

$$R^{\alpha}_{ijk} + R^{\alpha}_{jki} + R^{\alpha}_{kij} = 0 .$$

Covariant curvature tensor

The Riemannia-christoffel curvature tensor R^{α}_{ijk} is given by

$$R^{\alpha}_{ijk} = -\frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^k} + \frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^j} - \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k} + \Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j}$$

$$\therefore \quad R_{hijk} = g_{\alpha h} R^{\alpha}_{ijk}$$

$$= g_{\alpha h}\left(-\frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^k} + \frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^j} - \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k} + \Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j}\right)$$

$$= -g_{\alpha h}\frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^k} + g_{\alpha h}\frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^j} - g_{\alpha h}\Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k}$$
$$+ g_{\alpha h}\Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j}$$

$$= -\frac{\partial}{\partial x^k}\left(g_{\alpha h}\Gamma^{\alpha}_{ij}\right) + \Gamma^{\alpha}_{ij}\frac{\partial g_{\alpha h}}{\partial x^k} +$$
$$\frac{\partial}{\partial x^j}\left(g_{\alpha h}\Gamma^{\alpha}_{ik}\right) - \Gamma^{\alpha}_{ik}\frac{\partial g_{\alpha h}}{\partial x^j}$$
$$- \Gamma^{\beta}_{ij}\left(g_{\alpha h}\Gamma^{\alpha}_{\beta k}\right) + \Gamma^{\beta}_{ik}\left(g_{\alpha h}\Gamma^{\alpha}_{\beta j}\right)$$

$$= -\frac{\partial}{\partial x^k}\left(\Gamma_{ij,h}\right) + \Gamma^{\alpha}_{ij}\frac{\partial g_{\alpha h}}{\partial x^k} + \frac{\partial}{\partial x^j}\left(\Gamma_{ik,h}\right)$$
$$- \Gamma^{\alpha}_{ik}\frac{\partial g_{\alpha h}}{\partial x^j} - \Gamma^{\beta}_{ij}\Gamma_{\beta k,h} + \Gamma^{\beta}_{ik}\Gamma_{\beta j,h}$$

$$= -\frac{\partial}{\partial x^k}\left(\Gamma_{ij,h}\right) + \frac{\partial}{\partial x^j}\left(\Gamma_{ik,h}\right) + \Gamma^{\alpha}_{ij}\left(\frac{\partial g_{\alpha h}}{\partial x^k} - \Gamma_{\alpha k,h}\right.$$
$$\left. - \Gamma^{\alpha}_{ik}\left(\frac{\partial g_{\alpha h}}{\partial x^j} - \Gamma_{\alpha j,h}\right)\right).$$

$$= -\frac{\partial}{\partial x^k}\left[\frac{1}{2}\left(\frac{\partial g_{hj}}{\partial x^i} + \frac{\partial g_{ih}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^h}\right)\right]$$

$$+ \frac{\partial}{\partial x^j}\left[\frac{1}{2}\left(\frac{\partial g_{hk}}{\partial x^i} + \frac{\partial g_{ih}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^h}\right)\right]$$

$$+ \Gamma^{\alpha}_{ij}\Gamma_{hk,\alpha} - \Gamma^{\alpha}_{ik}\Gamma_{hj,\alpha}$$

$$= \frac{1}{2}\left(-\frac{\partial^2 g_{hj}}{\partial x^k \partial x^i} - \frac{\partial^2 g_{ih}}{\partial x^k \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^k \partial x^h} + \frac{\partial^2 g_{hk}}{\partial x^j \partial x^i}\right.$$

$$\left. + \frac{\partial^2 g_{ih}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^h}\right) + \Gamma^{\alpha}_{ij}\Gamma_{hk,\alpha} - \Gamma^{\alpha}_{ik}\Gamma_{hj\alpha}$$

$$\therefore R_{hijk} = \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} + \frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{hj}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j}\right)$$

$$+ g_{\alpha\beta}\Gamma^{\alpha}_{ij}\Gamma^{\beta}_{hk} - g_{\alpha\beta}\Gamma^{\alpha}_{hj}\Gamma^{\beta}_{ik}.$$

$$\boxed{\begin{aligned} &-\Gamma^{\alpha}_{ik}\, g_{\alpha\beta}\Gamma^{\beta}_{hj} \\ &= -g_{\alpha\beta}\Gamma^{\alpha}_{ik}\Gamma^{\beta}_{hj} \\ &= -g_{\beta\alpha}\Gamma^{\beta}_{ik}\Gamma^{\alpha}_{hj} \\ &= -g_{\alpha\beta}\Gamma^{\alpha}_{hj}\Gamma^{\beta}_{ik} \end{aligned}}$$

This is the covariant form of the Riemann-christoffel curvature tensor called ~~covar~~ the covariant curvature tensor or Riemann symbol of the first kind.

~·~·~

<u>properties:</u>

1) Anti-symmetric property

(a) $R_{hijk} = -R_{ihjk}$

(b) $R_{hijk} = -R_{hikj}$

2) Symmetric property.

$R_{hijk} = R_{jkhi}$

3) Cyclic property:

$$R_{hijk} + R_{hjki} + R_{hkij} = 0$$