3) Cyclic property:

$$R_{hijk} + R_{hjki} + R_{hkij} = 0 .$$

✽ Bianchi Identities:

Statement: $$R^{\alpha}_{ijk,l} + R^{\alpha}_{ikl,j} + R^{\alpha}_{ilj,k} = 0 .$$

proof: The Riemann-christoffel curvature tensor R^{α}_{ijk} is given by

$$R^{\alpha}_{ijk} = -\frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^k} + \frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^j} - \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k} + \Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j}$$

Since at the pole of a geodesic co-ordinate system, both kinds of christoffel's symbols vanish, but not necessarily their derivatives, therefore at the pole P_0 of a geodesic coordinate system, we have

$$R^{\alpha}_{ijk} = -\frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^k} + \frac{\partial \Gamma^{\alpha}_{ik}}{\partial x^j} \longrightarrow (1)$$

Taking x-covariant derivative of (1) we get

$$R^{\alpha}_{ijk,l} = -\frac{\partial^2 \Gamma^{\alpha}_{ij}}{\partial x^l \partial x^k} + \frac{\partial^2 \Gamma^{\alpha}_{ik}}{\partial x^l \partial x^j} . \text{ at } P_0 \longrightarrow (2)$$

Rotating the indices j, k, l cyclically, we get

$$R^{\alpha}_{ikl,j} = -\frac{\partial^2 \Gamma^{\alpha}_{ik}}{\partial x^j \partial x^l} + \frac{\partial^2 \Gamma^{\alpha}_{il}}{\partial x^j \partial x^k} \text{ at } P_0 \longrightarrow (3)$$

and $$R^{\alpha}_{ilj,k} = -\frac{\partial^2 \Gamma^{\alpha}_{il}}{\partial x^k \partial x^j} + \frac{\partial^2 \Gamma^{\alpha}_{ij}}{\partial x^k \partial x^l} \text{ at } P_0 \longrightarrow (4)$$

Adding (2), (3) and (4) we obtain

$$R^{\alpha}_{ijk,l} + R^{\alpha}_{ikl,j} + R^{\alpha}_{ilj,k} = 0 \text{ at } P_0 . \longrightarrow (5)$$

But P_0 is an arbitrary point in the Riemannian space, therefore, the relation (5) is true for all points of the space. Further since (5) is a tensor equation, therefore it must hold good in every coordinate system. Thus, the relation (5) is true for all points of Riemannian space and for all coordinate systems.

Hence $R^{\alpha}_{ijk,l} + R^{\alpha}_{ikl,j} + R^{\alpha}_{ilj,k} = 0$

are identities called Bianchi identities.

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Another form of Bianchi identities :→

Bianchi identities are given by

$$R^{\alpha}_{ijk,l} + R^{\alpha}_{ikl,j} + R^{\alpha}_{ilj,k} = 0 \longrightarrow (1)$$

Multiplying both sides by $g_{\alpha h}$, we get

$$g_{\alpha h} R^{\alpha}_{ijk,l} + g_{\alpha h} R^{\alpha}_{ikl,j} + g_{\alpha h} R^{\alpha}_{ilj,k} = 0 \longrightarrow (2)$$

Since the fundamental tensors are covariant constant, therefore (2) can be rewritten as

$$(g_{\alpha h} R^{\alpha}_{ijk})_{,l} + (g_{\alpha h} R^{\alpha}_{ikl})_{,j} + (g_{\alpha h} R^{\alpha}_{ilj})_{,k} = 0$$

$$\Rightarrow R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0$$

which is the covariant form of Bianchi identities.

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