Einstein Tensor

The Einstein tensor, usually denoted by G_{ij} and is defined by

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$$

where R_{ij} is the Ricci tensor; g_{ij} is the fundamental tensor and R is the Ricci scalar curvature.

Since R_{ij} and g_{ij} are symmetric tensors of rank two, therefore the Einstein tensor $R_{ij} - \frac{1}{2} g_{ij} R$ is also a symmetric tensor of rank two.

Raising the suffixes, we get

$$G^{ij} = R^{ij} - \frac{1}{2} g^{ij} R \quad \text{(contravariant form)}$$

Also
$$\left.\begin{aligned} G^i_j &= R^i_j - \frac{1}{2} g^i_j R \\ \text{and } G^j_i &= R^j_i - \frac{1}{2} g^j_i R \end{aligned}\right\} \text{(mixed form)}$$

Since $g^i_j = \delta^i_j$ (Kronecker delta), therefore we may also write

$$G^i_j = R^i_j - \frac{1}{2} \delta^i_j R \quad \text{and}$$

$$G^j_i = R^j_i - \frac{1}{2} \delta^j_i R$$

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Divergence of a tensor:

The divergence of a tensor is its contracted covariant derivative. (Recall that in first semester class, we have discussed about contraction of tensors and divergence of vectors).

The divergence of a vector A^i is defined as $\operatorname{div} A^i = A^i_{,i}$

Similarly, $\operatorname{div} A^{ij} = A^{ij}_{,i}$

or $\operatorname{div} A^{ij}_{,j}$

Also $\operatorname{div} A^i_j = A^i_{j,i}$

$\operatorname{div} A^j_i = A^j_{i,j}$ etc..

Contraction of Bianchi identities:

Bianchi identities are given by

$$R^\alpha_{ijk,l} + R^\alpha_{ikl,j} + R^\alpha_{ilj,k} = 0 \longrightarrow (1)$$

Applying antisymmetric property to the second term, we get

$$R^\alpha_{ijk,l} - R^\alpha_{ilk,j} + R^\alpha_{ilj,k} = 0 \longrightarrow (2)$$

Contraction of (2) w.r.to the indices α and k gives

$$R_{ij,l} - R_{il,j} + R^\alpha_{ilj,\alpha} = 0 \longrightarrow (3)$$

Taking inner product with g^{ih} we get

$$g^{ih} R_{ij,l} - g^{ih} R_{il,j} + g^{ih} R^\alpha_{ilj,\alpha} = 0 \longrightarrow (4)$$

Since the fundamental tensors are covariant constant, therefore from (4) we get

$$(g^{ih} R_{ij})_{,l} - (g^{ih} R_{il})_{,j} + (g^{ih} R^\alpha_{ilj})_{,\alpha} = 0$$

$$\Rightarrow R^h_{j,l} - R^h_{l,j} + (g^{ih} R^\alpha_{ilj})_{,\alpha} = 0 \longrightarrow (5)$$

But $g^{ih} R^\alpha_{ilj} = g^{ih}(g^{\alpha\beta} R_{\beta ilj})$

$$= g^{\alpha\beta}(g^{ih} R_{\beta ilj})$$