Topology and topological spaces:

Definition: Let X be a non-empty set. A collection $\mathcal{T}$ of subsets of X is said to be a topology on X if it has the following properties:

(i) ϕ and X are in $\mathcal{T}$.

(ii) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(iii) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

The ordered pair $(X, \mathcal{T})$ is then called a topological space. Properly speaking, a topological space is an ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X.

Examples:

(1) Let $X = \{a\}$, then $\mathcal{T} = \{\phi, X\}$ is a topology on X.

(2) Let $X = \{a, b, c\}$, then

$$\mathcal{T}_1 = \{\phi, X\}$$

$$\mathcal{T}_2 = \{\phi, \{a\}, X\}$$

$$\mathcal{T}_3 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$$

$$\mathcal{T}_4 = \{\phi, \{a\}, \{a, b\}, X\}$$

$$\mathcal{T}_5 = \{\phi, \{a\}, \{b, c\}, X\}$$

are some topologies on X.

(9) Let $X = \{a, b, c, d, e\}$.

Consider the following classes to subsets of X

$$\tau_1 = \{X, \phi, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$$

$$\tau_2 = \{\phi, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, X\}$$

$$\tau_3 = \{\phi, \{a\}, \{c,d\}, \{a,c,d\}, \{a,b,d,e\}\}$$

Then τ_1 is a topology on X, but τ_2 and τ_3 are not topology on X.

Discrete and Indiscrete topology:

Let X be a non empty set, the the collection of all subsets of X is a topology on X called the discrete topology on X. The collection consisting X and ϕ only is also a topology on X called the Indiscrete topology on trivial topology on X. Discrete topology is denoted by $\mathcal{D}$ and Indiscrete topology is denoted by $\mathcal{I}$.

Finite complement topology on co-finite topology:

Let X be a non empty set. Let,

$\tau = \{U : U \subseteq X$ and $X - U$ is either finite on is all of $X\}$

Then $\mathcal{T}$ is a topology on X called co-finite topology.

Pf: Here $X \neq \phi$ and

$\mathcal{T} = \{U : U \subseteq X$ and $X - U$ is either finite or is all of $X\}$

(i) $X - \phi = X$, Therefore $\phi \in \mathcal{T}$.

Also $X - X = \phi$, a finite set, Therefore $X \in \mathcal{T}$

(ii) Let $\{U_\alpha\}$ is any subcollection of $\mathcal{T}$.

$\therefore U_\alpha \in \mathcal{T}, \ \forall \alpha$

$\Rightarrow X - U_\alpha$ are finite or all of X, $\forall \alpha$

Now $X - U_\alpha$ are finite, $\forall \alpha$

$\Rightarrow \bigcap_\alpha (X - U_\alpha)$ is also finite.

$\Rightarrow X - \bigcup_\alpha U_\alpha$ is also finite.

$\Rightarrow \bigcup_\alpha U_\alpha \in \mathcal{T}$

$\therefore$ arbitrary union of members of $\mathcal{T}$ is in $\mathcal{T}$.

(iii) Let $\{U_1, U_2, \ldots, U_n\}$ is a finite subcollection of elements of $\mathcal{T}$.

$\therefore$ $X - U_i$ are finite on all of X, $\forall i = 1, 2, \ldots, n$

$\Rightarrow \bigcup_{i=1}^{n} (X - U_i)$ is finite,

$\Rightarrow X - \bigcap_{i=1}^{n} U_i$ is finite.

$\Rightarrow \bigcap_{i=1}^{n} U_i \in \mathcal{T}$

i.e finite intersection of members of $\mathcal{T}$ is in $\mathcal{T}$.

$\therefore$ $\mathcal{T}$ is a topology on X.

Comparable topology:

Let T_1 and T_2 be two topology on a non empty set X. If $T_1 \supset T_2$, then T_1 is finer or stronger or larger than T_2 and T_2 is coarser or weaker or smaller than T_1. T_1 is comparable with T_2 if either $T_1 \supset T_2$ or $T_1 \subset T_2$.

Ex: Let, $X = \{a, b, c\}$

$T_1 = \{\phi, X\}$

$T_2 = \{\phi, \{a\}, \{b, c\}, X\}$

$T_3 = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$

$T_4 = \{\phi, \{a\}, X\}$

Then $T_1 \subset T_2$, $T_1 \subset T_3$, $T_1 \subset T_4$

$\therefore$ T_1 and T_2, T and T_3 and, T_1 and T_4 are comparable.

But T_2 and T_4 are not comparable. Also T_3, T_4 are not comparable.