where τ_{ij} is called viscous stress components. All real fluids are compressible and viscous. However, the characteristic of the fluid very widely in different cases.

#

2.4. Stokesian fluid, Newtonian fluid:

It is generally assumed viscous stress matrix in the function of a deformed matrix. If the functional relation is a non-linear then it is expressed as

$$\tau_{ij} = f_{ij}\left[D_{pq}\right]$$

And, in this case, the fluid is called stokesion fluid or non-Newtonian fluid. When the function is a linear one of the form

$$\tau_{ij} = k_{ij} D_{pq}, \quad D_{pq}, \text{ the deformation matrix.}$$

where K_{ij} is a constant, then the fluid is called a Newtonian fluid. For a Newtonian fluid it is assumed that

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$= -p\delta_{ij} + \lambda\,\delta_{ij} D_{KK} + 2\mu D_{ij}$$

where λ and μ are called the material constant of the fluid. From this relation, we get, (Putting $i=j$)

$$\sigma_{ii} = -p\delta_{ii} + \lambda\,\delta_{ii} D_{KK} + 2\mu D_{ii}$$

$$= -3p + 3\lambda D_{KK} + 2\mu D_{KK}$$

$$\Rightarrow \tfrac{1}{3}\sigma_{ii} = -p + k^* D_{kk}, \quad \text{where } D_{KK} = \lambda + \tfrac{2}{3}\mu, \text{ Bulk viscosity.}$$

The condition that k^* is equal to zero i.e. Bulk viscosity is zero, called stoke's condition. This condition states that the pressure p is defined as the avarage of the normal stress at the point within the fluid. Assuming that $K^{*}=0$, we find $\lambda = -\frac{2}{3}\mu$.

Therefore,

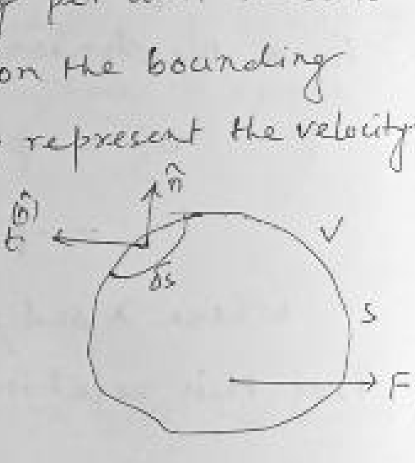
$$\sigma_{ij} = -p\delta_{ij} + 2\mu\left[-\frac{1}{3}\delta_{ij} D_{kk} + D_{ij}\right]$$

is the form of the stress matrix for Newtonian fluid. We deduce Navier-stokes equation with the above form of σ_{ij}.

#

2.5. Equation of motion (Linear momentum principle).

Suppose an amount of moving fluid is occupying a volume V at time t. External (Body) forces acting per unit mass is $\vec{F}$ within it and surface force acting on the bounding surface is $t^{(\hat{n})}$. ρ is the dencity and q_i represent the velocity within the fluid.



Newton's 2nd law states that the time rate of change of linear momentum of any portion of the medium is equal to the resultant force acting upon the portion considered.

Therefore, if the internal forces between the particles of the medium obeys the Newton third law of action and reaction then the momentum principle for the mass system is given by

$$\frac{D}{Dt}\int_V \rho q_i \, dv = \int_V \rho F_i \, dv + \int_S t_i^{(\hat{n})} \, ds$$

But $t_i^{(\hat{n})} = \sigma_{ji} n_j$

$$\therefore \frac{D}{Dt}\int_V \rho q_i \, dv = \int_V \rho F_i \, dv + \int_S \sigma_{ji} n_j \, ds$$

$$\Rightarrow \frac{D}{Dt}\int_V \rho q_i \, dv = \int_V \rho F_i \, dv + \int_V \frac{\partial \sigma_{ji}}{\partial x_j} \, dv$$

$$\Rightarrow \int_V \frac{Dq_i}{Dt} \rho \, dv = \int_V \rho F_i \, dv + \int_V \frac{\partial \sigma_{ji}}{\partial x_j} \, dv$$

$$\Rightarrow \int_V \left(\frac{Dq_i}{Dt}\rho - \rho F_i - \frac{\partial \sigma_{ji}}{\partial x_j} \right) dv = 0$$

Since V is arbitrary, therefore the integrand must vanish and hence

$$\rho \frac{Dq_i}{Dt} - \rho F_i - \frac{\partial \sigma_{ji}}{\partial x_j} = 0$$

at every point within V, $i, j = 1, 2, 3$

This is known as the Equation of motion of the fluid medium at every point. It can be shown that stress matrix is a symmetric matrix. Therefore, the moment of momentum principle does not give a new equation. Hence

$$\rho \frac{Dq_i}{Dt} = \rho F_i + \frac{\partial \sigma_{ji}}{\partial x_j} \;, \; i, j = 1, 2, 3 \longrightarrow (1) \quad (\sigma_{ij} = \sigma_{ji})$$

gives the equation of motion.

Now, for perfect fluid we know that

$$\sigma_{ij} = -p\,\delta_{ij}$$

From (1), we have,

$$\rho \frac{Dq_i}{Dt} = \rho F_i + \frac{\partial}{\partial x_j}\left(-p\delta_{ij}\right)$$

$$= \rho F_i + \frac{\partial}{\partial x_i}\left(-p\right)$$

$$\therefore \rho \frac{Dq_i}{Dt} = \rho F_i - \frac{\partial p}{\partial x_i}$$

is called Euler's equation of motion.

Multiplying by unit vectors and then, adding we have,

$$\rho \frac{D\vec{q}}{Dt} = \rho \vec{F} - \vec{\nabla} p$$

where $\sigma_{ij} = -p\delta_{ij} + 2\mu\left[-\frac{1}{3}\delta_{ij} D_{KK} + D_{ij}\right]$

then from (1), we have,

$$\rho \frac{D\vec{q}_i}{Dt} = \rho F_i + \frac{\partial}{\partial x_j}\left[-p\delta_{ij} + 2\mu\left\{-\frac{1}{3}\delta_{ij} D_{KK} + D_{ij}\right\}\right]$$

$$= \rho F_i - \frac{\partial p}{\partial x^i} - \frac{2\mu}{3}\frac{\partial}{\partial x_i}\left(D_{KK}\right) + 2\mu \frac{\partial}{\partial x_j}\left(D_{ij}\right)$$

But $D_{ij} = \frac{1}{2}\left[\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x^i}\right]$

$$= \frac{\partial q_i}{\partial x_i}$$

$$= \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3}$$

$$= \operatorname{div} \vec{q}$$

$$= \theta.$$

$$\therefore \rho \frac{Dq_i}{Dt} = \rho F_i - \frac{\partial p}{\partial x_i} - \frac{2\mu}{3}\frac{\partial}{\partial x_i}\theta + \frac{2\mu}{2}\frac{\partial}{\partial x_j}\left[\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right]$$

$$= \rho F_i - \frac{\partial p}{\partial x_i} - \frac{2\mu}{3}\frac{\partial \theta}{\partial x_i} + \mu\left[\frac{\partial^2 q_i}{\partial x_j \partial x_j} + \frac{\partial^2 q_j}{\partial x_i \partial x_j}\right]$$

$$= \rho F_i - \frac{\partial p}{\partial x_i} - \frac{2\mu}{3}\frac{\partial \theta}{\partial x_i} + \mu\left[\left\{\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right\} q_i + \frac{\partial}{\partial x_i}\left\{\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3}\right\}\right]$$

$$\Rightarrow \rho \frac{Dq_i}{Dt} = \rho F_i - \frac{\partial p}{\partial x_i} - \frac{2\mu}{3}\frac{\partial \theta}{\partial x_i} + \mu \nabla^2 q_i + \mu \frac{\partial \theta}{\partial x_i}$$

$$= \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\mu}{3}\frac{\partial \theta}{\partial x_i} + \mu \nabla^2 q_i$$

$$\Rightarrow \frac{Dq_i}{Dt} = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\nu}{3}\frac{\partial \theta}{\partial x_i} + \nu \nabla^2 q_i$$

where $\nu = \frac{\mu}{\rho}$ is kinematic viscosity. This equation is called Navier-stoke's equation of motion.

for incompressible fluid, $\operatorname{div} \vec{q} = 0$ i.e. $\theta = 0$ and hence, Navier-stoke's equation reduces to

$$\frac{Dq_i}{Dt} = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 q_i$$

In vector form, the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{q}) = 0$$

and the equation of motion is

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p + \frac{\nu}{3}\nabla \theta + \nu \nabla^2 \vec{q}$$

or, $\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{q}$ (for incompressible fluid)

The equation of continuity and the equation of motion comparise four equation for four unknowns.

$\vec{q} = (q_1, q_2, q_3)$ and p.

Cor(1) The Navier-Stoke's equation of motion in cartesian coordinates are given by the set of equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w .$$

where $\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$, $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$

(2) Since, $\nabla\times(\nabla\times q) = \nabla(\nabla\cdot\vec{q}) - \nabla^2\vec{q}$

and for incompressible fluid $\nabla\cdot\vec{q} = 0$

Therefore, $\nabla\times(\nabla\times\vec{q}) = -\nabla^2\vec{q}$

Hence, Navier-Stoke's equation of motion can also be expressed as

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p - \nu(\nabla\times \mathrm{curl}\,\vec{q}) .$$

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