2.7. Equation of vorticity for incompressible fluid:

Navier-Stoke's equation of motion is

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p + \frac{\nu}{3}\nabla\theta + \nu\nabla^2\vec{q} \longrightarrow (1), \text{ where } \theta = \operatorname{div}\vec{q}.$$

Let us assume that

(i) the external force $\vec{F}$ be conservative and be derivable from a scalar function Ω so that $\vec{F} = -\nabla\Omega$.

(ii) p and ρ are connected

(iii) ν or μ is constant.

then under the conditions (i) and (ii) we have,

$$\vec{F} - \frac{1}{\rho}\nabla p = -\nabla\Omega - \nabla\int\frac{dp}{\rho}, \text{ where } \frac{1}{\rho}\nabla p = \nabla\int\frac{dp}{\rho}$$

$$= -\nabla\left(\Omega + \int\frac{dp}{\rho}\right)$$

Equation (1) can be written as

$$\frac{D\vec{q}}{Dt} = -\nabla\left[\Omega + \int\frac{dp}{\rho}\right] + \frac{\nu}{3}\nabla\theta + \nu\nabla^2\vec{q}.$$

Taking curl of this equation,

$$\operatorname{curl}\frac{D\vec{q}}{Dt} = -\operatorname{curl}\operatorname{grad}\left[\Omega + \int\frac{dp}{\rho}\right] + \operatorname{curl}\frac{\nu}{3}\nabla\theta + \operatorname{curl}\nu\nabla^2\vec{q} \rightarrow (ii)$$

But $\dfrac{D\vec{q}}{Dt} = \dfrac{\partial\vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}$

$$= \frac{\partial\vec{q}}{\partial t} + \nabla\frac{1}{2}\left(\vec{q}^2\right) - \vec{q}\times\operatorname{curl}\vec{q}$$

$$= \frac{\partial\vec{q}}{\partial t} - \vec{q}\times\vec{\zeta} + \operatorname{grad}\frac{1}{2}(q^2), \text{ where } \vec{\zeta} = \operatorname{curl}\vec{q}$$

$$\therefore \text{curl}\frac{D\vec{q}}{Dt} = \text{curl}\frac{\partial\vec{q}}{\partial t} - \nabla\times(\vec{q}\times\vec{\zeta}) + \text{curl grad}\,\frac{1}{2}(q^2).$$

$$= \text{curl}\frac{\partial\vec{q}}{\partial t} - \nabla\times(\vec{q}\times\vec{\zeta}).$$

Therefore equation (II) becomes

$$\nabla\times\frac{\partial\vec{q}}{\partial t} - \nabla\times(\vec{q}\times\vec{\zeta}) = \nabla\times\left(\frac{\nu}{3}\nabla\theta\right) + \nabla\times(\nu\nabla^2\vec{q})$$

$$\Rightarrow \frac{\partial}{\partial t}(\nabla\times\vec{q}) - (\vec{\zeta}\cdot\nabla)\vec{q} + (\vec{q}\cdot\nabla)\vec{\zeta} + \vec{\zeta}(\nabla\cdot\vec{q}) - \vec{q}(\nabla\cdot\vec{\zeta})$$

$$= \frac{\nu}{3}\nabla\times(\nabla\theta) + \nu\nabla\times(\nabla^2\vec{q})$$

[Since curl is commutative with $\frac{\partial}{\partial t}$ and ∇ and $\nabla\times(\vec{q}\times\vec{\zeta}) =$]

$$\Rightarrow \frac{\partial\vec{\zeta}}{\partial t} + (\vec{q}\cdot\nabla)\vec{\zeta} - (\vec{\zeta}\cdot\nabla)\vec{q} + \vec{\zeta}(\nabla\cdot\vec{q}) = \nu\nabla^2\vec{\zeta}$$

$$\Rightarrow \frac{D\vec{\zeta}}{Dt} - (\vec{\zeta}\cdot\nabla)\vec{q} + \vec{\zeta}(\nabla\cdot\vec{q}) = \nu\nabla^2\cancel{\vec{q}}\,\vec{\zeta}$$

$$\Rightarrow \frac{D\vec{\zeta}}{Dt} - (\vec{\zeta}\cdot\nabla)\vec{q} + \vec{\zeta}\left(-\frac{1}{\rho}\frac{D\rho}{Dt}\right) = \nu\nabla^2\vec{\zeta} \;\; ; \left(\frac{1}{\rho}\frac{D\rho}{Dt} + \text{div}\,\vec{q} = 0\right) \longrightarrow (III)$$

For incompressible (fluid) flow $\frac{D\rho}{Dt} = 0$ and (III) reduces to

$$\frac{D\vec{\zeta}}{Dt} - (\vec{\zeta}\cdot\nabla)\vec{q} = \nu\nabla^2\vec{\zeta} \quad \longrightarrow (IV)$$

Equations (III) and (IV) are called the vorticity equations after the name of Helmholtz.

If the motion is slow, then $(\vec{q}\cdot\nabla)\vec{\zeta}$ and $(\vec{\zeta}\cdot\nabla)\vec{q}$ are negligible quantity then equation of vorticity for incompressible flow is

$$\frac{D\vec{\zeta}}{Dt} = \nu\nabla^2\vec{\zeta}$$

$$\Rightarrow \frac{D\vec{\zeta}}{\partial t} + (\vec{q}\cdot\nabla)\vec{\zeta} = \nu\nabla^2\vec{\zeta}$$

$$\Rightarrow \frac{\partial\vec{\zeta}}{\partial t} = \nu\nabla^2\vec{\zeta} \quad (\text{approx.})$$

This is the equation for heat. It shows that vorticity $\vec{\zeta}$ behave like temperature, when the motion is slow. As heat flows from some source, so also vorticity should come from same source. Generally, immersed solid bodies are the source of vorticity.

2.8. Rate of change of circulation of viscous fluid:

If $\vec{q}$ be the vorticity and $d\vec{s}$ be the curve length at the point on a closed reducible curve C, then $\oint_C \vec{q}\cdot d\vec{s}$ is called the circulation. It can be shown that for perfect fluid

$\frac{D}{Dt}$(circulation along c) is nil and for viscous fluid it depends on viscosity constant and vorticity.

Let $K = \oint_C \vec{q}\cdot d\vec{s}$ be the circulation along the closed reducible curve C. Now, the rate of change of circulation when the fluid moves is

$$\frac{DK}{Dt} = \frac{D}{Dt}\oint_C \vec{q}\cdot d\vec{s}$$

$$= \oint_C \frac{D\vec{q}}{Dt}\cdot d\vec{s} + \oint_C \vec{q}\cdot\frac{D}{Dt}(d\vec{s}) \longrightarrow (1)$$

Now, $$\frac{D}{Dt}(d\vec{s}) = \frac{D}{Dt}\left[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right]$$

$$= \frac{D}{Dt}\left[d(\hat{i}x + \hat{j}y + \hat{k}z)\right]$$

$$= d\left[\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(\hat{i}x + \hat{j}y + \hat{k}z)\right]$$

$$= d\left[\hat{i}u + \hat{j}v + \hat{k}w\right]$$

$$= d\vec{q}$$

$$= \frac{\partial \vec{q}}{\partial s}\, ds \quad , \quad ds = |d\vec{s}|$$

From (1),

$$\frac{DK}{Dt} = \oint_c \frac{D\vec{q}}{Dt} \cdot d\vec{s} + \oint_c \vec{q} \cdot \frac{\partial \vec{q}}{\partial s}\, ds .$$

$$= \oint_c \frac{D\vec{q}}{Dt} \cdot d\vec{s} + \left[\frac{1}{2} q^2\right]_c$$

$$= \oint_c \frac{D\vec{q}}{Dt} \cdot d\vec{s} \longrightarrow (\text{ii})$$

Now, from Navier-stokes theorem we have,

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{q} + \frac{\nu}{3} \nabla \theta$$

$\therefore$ Substituting in (ii), we have,

$$\frac{DK}{Dt} = \oint_c \left[\vec{F} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{q} + \frac{\nu}{3} \nabla \theta\right] \cdot d\vec{s} = 0 \longrightarrow (\text{iii})$$

Now, if the external force $\vec{F}$ ~~becomes~~ conservative and be derivable from a single valued potential i.e. If

(1) $\vec{F} = -\nabla \Omega$

(2) If the pressure p is a function of density ρ and

(3) ν is a constant,

then from (1) & (2), we can find

$$\vec{F} - \frac{1}{\rho} \nabla p = -\nabla\left[\Omega + \int \frac{dp}{\rho}\right]$$

$$\therefore \frac{DK}{Dt} = \oint_c \left[-\nabla\left(\Omega + \int \frac{dp}{\rho}\right) + \frac{\nu}{3} \nabla \theta + \nu \nabla^2 \vec{q}\right] \cdot d\vec{s} \longrightarrow (\text{iv})$$

But $\nabla s \cdot d\vec{s} = \frac{\partial s}{\partial s} ds$, s is a scalar function and $ds = |d\vec{s}|$

From (IV)

$$\frac{DK}{Dt} = \oint_C -\frac{\partial}{\partial s}\left(-\Omega + \int \frac{dp}{\rho}\right) ds + \frac{\nu}{3}\oint_C \frac{\partial}{\partial s}\theta\, ds + \oint_C \nu \nabla^2 \vec{q}\cdot d\vec{s}$$

$$= 0 + 0 + \oint_C \nu \nabla^2 \vec{q}\cdot d\vec{s}$$

$$= \nu \oint_C \nabla^2 \vec{q}\cdot d\vec{s} \longrightarrow (V)$$

Again, $\nabla\times(\nabla\times\vec{q}) = -\nabla^2\vec{q} + \nabla(\nabla\cdot\vec{q})$

For incompressible flow, $\nabla\cdot\vec{q} = 0$, then

$$\nabla\times(\nabla\times\vec{q}) = -\nabla^2\vec{q}$$

$$\Rightarrow \nabla^2\vec{q} = -\nabla\times\vec{\xi} \text{ , where } \vec{\xi} = \nabla\times\vec{q} = \text{curl}\,\vec{q}.$$

Thus (V) can be written as

$$\frac{DK}{Dt} = -\nu \oint_C \text{curl}\,\vec{\xi}\cdot d\vec{s}$$

Now, if $\nu = 0$, $\frac{DK}{Dt} = 0$

If $\nu \neq 0$, $\frac{DK}{Dt}$ depends on ν on $\vec{\xi}$.

By using stoke's theorem, we can write,

$$\oint_C \vec{q}\cdot d\vec{s} = \iint_\Gamma \vec{\xi}\cdot\vec{n}\, ds$$

where Γ be the surface with C as boundary.

$$\oint_C \vec{q}\cdot d\vec{s} = \iint_\Gamma \vec{\xi}\cdot\vec{n}\, ds$$

= Flux of vorticity or vortex line through C and cutting Γ at odd number of pts.

= constant.

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