The divergence of a vector A^i is defined as $\text{div}\, A^i = A^i_{,i}$

Similarly, $\text{div}\, A^{ij} = A^{ij}_{,i}$

or $\text{div}\, A^{ij}_{,j}$

Also $\text{div}\, A^i_j = A^i_{j,i}$

$\text{div}\, A^j_i = A^j_{i,j}$ etc..

contraction of Bianchi identities:

Bianchi identities are given by

$$R^\alpha_{ijk,l} + R^\alpha_{ikl,j} + R^\alpha_{ilj,k} = 0 \longrightarrow (1)$$

Applying antisymmetric property to the second term, we get

$$R^\alpha_{ijk,l} - R^\alpha_{ilk,j} + R^\alpha_{ilj,k} = 0 \longrightarrow (2)$$

contraction of (2) w.r.to the indices α and k gives

$$R_{ij,l} - R_{il,j} + R^\alpha_{ilj,\alpha} = 0 \longrightarrow (3)$$

Taking inner product with g^{ih} we get

$$g^{ih} R_{ij,l} - g^{ih} R_{il,j} + g^{ih} R^\alpha_{ilj,\alpha} = 0 \longrightarrow (4)$$

Since the fundamental tensors are covariant constant, therefore from (4) we get

$$(g^{ih} R_{ij})_{,l} - (g^{ih} R_{il})_{,j} + (g^{ih} R^\alpha_{ilj})_{,\alpha} = 0$$

$$\Rightarrow R^h_{j,l} - R^h_{l,j} + (g^{ih} R^\alpha_{ilj})_{,\alpha} = 0 \longrightarrow (5)$$

But $g^{ih} R^\alpha_{ilj} = g^{ih} (g^{\alpha\beta} R_{\beta ilj})$

$$= g^{\alpha\beta} (g^{ih} R_{\beta ilj})$$

$$= g^{\alpha\beta}(-g^{ih} R_{i\beta l j})$$

$$= -g^{\alpha\beta} R^{h}_{\beta l j}$$

$$= g^{\alpha\beta} R^{h}_{\beta j l}$$

Therefore (5) becomes

$$R^{h}_{j,l} - R^{h}_{l,j} + (g^{\alpha\beta} R^{h}_{\beta j l})_{,\alpha} = 0$$

contracting w. r. to the indices h and l, we get

$$R^{h}_{j,h} - R^{h}_{h,j} + (g^{\alpha\beta} R^{h}_{\beta j h})_{,\alpha} = 0.$$

$$\Rightarrow R^{h}_{j,h} - R_{,j} + (g^{\alpha\beta} R_{\beta j})_{,\alpha} = 0$$

$$\Rightarrow R^{h}_{j,h} - R_{,j} + R^{\alpha}_{j,\alpha} = 0$$

$$\Rightarrow R^{i}_{j,i} - R_{,j} + R^{i}_{j,i} = 0$$

$$\Rightarrow 2R^{i}_{j,i} - (g^{i}_{j} R)_{,i} = 0$$

$$\Rightarrow (R^{i}_{j} - \tfrac{1}{2} g^{i}_{j} R)_{,i} = 0$$

$$\Rightarrow \cancel{G^{i}_{j,i}}\ G^{i}_{j,i} = 0 \longrightarrow (6)$$

where $G^{i}_{j} = R^{i}_{j} - \frac{1}{2} g^{i}_{j} R$ is the Einstein tensor.

The equation (6) shows that the divergence of the Einstein tensor is identically zero.

Consequently, the Einstein tensor is divergence free.

Note: (i) If the divergence of a tensor is identically zero, then we say that the tensor is divergence free.

(ii) Contraction of a Bianchi identities leads to the fact that the Einstein tensor is divergence free. It is a very important result in the General Theory of Relativity.

Flat Space-time

A space-time is said to be flat if it is possible to construct in it a Galilian frame of reference, i.e. if a coordinate system can be found in it for which the fundamental tensors g_{ij} are constants.

e.g.: the space-time given by the metric

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$ is

flat since it can be transformed to $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$.